Exercises 10

April 7, 2021

1 Heat Equation

Date: **Apr 7, 2021** Deadline: **Apr 26, 2021**

1.1 1)

Check whether the superposition principle holds in the following examples of PDE problems. That is, assuming that $u(x,t)$ and $v(x,t)$ are solutions of the given problems, you have to check whether $u+v$ and $c \cdot u$ are still solutions. Note that when boundary conditions are specified you have to check whether **both** the equation and the boundary conditions are satisfied.

a) $\frac{\partial^3 u}{\partial t^3} = x^2 \frac{\partial^2 u}{\partial x^2}$.

b) $\frac{\partial u}{\partial t} = u \frac{\partial u}{\partial x}$.

c) $\frac{\partial^2 u}{\partial t \partial x} = t \frac{\partial u}{\partial t}$ with boundary conditions $u(0,t) = 0$, $\frac{\partial u}{\partial x}(1,t) = 0$.

d) $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(0,t) = 0, u(1,t) = 5t$.

2 2)

a) Find the solution to the heat equation (with $c = 1$)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

on the interval $[0, L]$ satisfying homogeneous Dirichlet boundary condition (that is, $u(0,t) = 0$ and $u(L,t) = 0$ for all t) and with initial datum

$$u(x,0) = 4\sin(\frac{5\pi x}{L}) + 7\sin(\frac{11\pi x}{L}).$$

b) Find the solution to the non-homogeneous heat equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 1$$

satisfying the same boundary and initial conditions as in part **a)**.

Hint: Consider $v(x,t) = u(x,t) + x(x-L)/2$.

2.1 3)

The goal of this problem is to solve the heat equation with mixed boundary conditions. You will do it in steps.

a) Show that for n, m positive integers

$$\int_0^\pi \cos\left(\left(n+\frac{1}{2}\right)x\right)\cos\left(\left(m+\frac{1}{2}\right)x\right)dx = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{cases} .$$

It is helpful to remember trigonometric formulas here.

b) Solve the heat equation ($c = 1$)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = u(\pi,t) = 0 \quad \text{for all } t$$

and initial condition

$$u(x,0) = \begin{cases} x & 0 \leq x \leq \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} \leq x \leq \pi \end{cases} .$$

To do this, repeat the separation of variables approach discussed in the lectures; you should use part **a)** at some point. Note that the meaning of part **a)** is finding an orthonormal basis for $[0,\pi]$ that satisfies mixed boundary conditions.

2.2 4)

The goal of this problem is to find the steady state temperature in a thin square plate. Find the solution to the following Dirichlet problem in the square $[0,a] \times [0,a]$:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

with boundary conditions

$$\begin{aligned} u(x,0) &= 20 \quad 0 \leq x < a \\ u(x,a) &= 20 \quad 0 \leq x < a \\ u(0,y) &= 20 \quad 0 \leq y \leq a \\ u(a,y) &= 90 \quad 0 \leq y \leq a. \end{aligned}$$

Hint: Subtract first the right constant to have 0 boundary conditions on 3 sides of the square, then use separation of variables as discussed in Section 12.6 of Kreyszig. Note that the roles of x and y are interchanged here compared to the example in the book.

Extra) What is the steady state temperature if the boundary condition on the right side of the square is changed to $u(a, y) = 20$ (and the others are left unchanged)? No calculation is required here, just a moment of thought.

2.3 5)

a) Plot the heat kernel

$$\Phi^t(x) = \frac{1}{2\sqrt{\pi t}} e^{-\frac{x^2}{4t}}$$

for $x \in [-1, 1]$ for $t = 1, t = 1/10, t = 1/100$.

b) The error function is a very common function in many computations in physics and engineering. It is not an elementary function, and is defined through an integral:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy.$$

Show that erf is an odd function, that is $erf(-x) = -erf(x)$.

c) Find the solution to the heat equation (with $c = 1$) on the real line with initial datum

$$u(x, 0) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

and write it using erf.

d) Express in integral form the solution to the heat equation (with $c = 1$) on the real line with initial datum

$$u(x, 0) = \frac{\sin x}{x}.$$

You may use the following fact: the Fourier cosine integral of the function

$$g(p) = \begin{cases} 1 & 0 < p < 1 \\ 0 & p > 1 \end{cases}$$

is

$$H(v) = \frac{2}{\pi} \frac{\sin v}{v}.$$