Norwegian University of Science and Technology
Department for Mathematical Science

Exercise set 11
Wave equation

Deadline: Friday 23 April

Mandatory exercises

1 In the following exercise you are asked to work out the details of Section "Løsning for fløyte" from Morten's Lecture Notes on the wave equation:

A standing pressure wave in a flute of length L can be described by the wave equation

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t), \quad x \in (0,L), \quad t > 0, \tag{1}$$

together with the Neumann boundary conditions

$$\partial_x u(0,t) = \partial_x u(L,t) = 0, \quad t > 0, \tag{2}$$

and the initial conditions

$$u(x,0) = f(x), \quad \partial_t u(x,0) = g(x), \quad x \in (0,L). \tag{3}$$

Then the solution to the problem (1)–(3) is

$$u(x,t) = A + \sum_{n=0}^{\infty} \Big(A_n \cos c\frac{n\pi}{L} t + B_n \sin c\frac{n\pi}{L} t \Big) \cos \frac{n\pi}{L} x, \tag{4}$$

where A is an arbitrary constant, and

$$A_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi}{L} x \, \mathrm{d}x, \tag{5}$$

and

$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \cos \frac{n\pi}{L} x \, \mathrm{d}x. \tag{6}$$

a) The vibrations in a flute are solutions to the problem (1)–(2), where $c = 343 m/s$ is the speed of sound at atmospheric pressure, and L is the length of the flute. Find all the solutions of the form $F_n(x)G_n(t)$. What is the deepest frequency a flute of length 70 cm can create?

b) Similar to our derivation of the solution representation for the wave, equation with homogeneous Dirichlet boundary conditions, use the technique of separation of variables and provide a detailed derivation of the solution representation (4)–(6).

2 Use the solution representation derived in the lectures and compute the solution to the wave equation on a bounded interval $I = (0, 1)$:

$$\partial_t^2 u(x,t) = 16\partial_x^2 u(x,t), \quad x \in (0,1), t > 0, \tag{7}$$

together with the Dirichlet boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t > 0, \tag{8}$$

and the initial conditions

$$u(x,0) = 1 - |2x-1|, \quad \partial_t u(x,0) = x(1-x) \quad \text{for } x \in (0,1). \tag{9}$$

3 Use the solution representation derived in the lectures and compute the solution to the wave equation on the entire real line $\mathbb{R}$

$$\partial_t^2 u(x,t) = 4\partial_x^2 u(x,t), \quad \text{for } x \in \mathbb{R}, t > 0, \tag{10}$$

together with the initial conditions

$$u(x,0) = \tanh(x^2), \quad \partial_t u(x,0) = \sin(2x), \quad x \in \mathbb{R}. \tag{11}$$

Recommended exercises

4 A clarinet is essentially closed in one end (in contrast with a flute, which is essentially open in both ends). Therefore, the standing waves in a clarinet satisfy the wave equation (1) together with the boundary conditions

$$u(0,t) = u_x(L,t) = 0, \quad t > 0. \tag{12}$$

a) Repeat exercise 1a) for the clarinet. Assume that the clarinet has the length 70 cm.

b) Find a solution representation for the wave equation with the boundary conditions above.