NTNU
Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4125 Matematikk 4N**

Solution

Academic contact during examination:

Phone:

Examination date: May 10, 2022

Examination time (from–to): 09:00–13:00

Permitted examination support material: C.
One sheet A4 paper, approved by the department (yellow sheet, "gul ark") with own handwritten notes.
Certain simple calculators.

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- Good Luck! | Lykke til! | Viel Glück!

Language: English

Number of pages: 22

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

In the exam one could obtain 100 points and the exam was graded using the usual grading scheme, i.e.

A	B	C	D	E	F
100–89	88–77	76–65	64–53	52–41	40 and less

And the grades are distributed as follows, where we split those that handed in an empty exam

A	B	C	D	E	F	empty	$\sum$
6 2.1 %	16 5.6 %	50 17.6 %	55 19.4 %	71 25 %	72 25.4 %	14 4.9%	284

Problem 1. (Polynomial interpolation, 8 points)

Find the polynomial $p(x)$ of lowest possible degree that interpolates the following values.

x_i	-2	-1	0	1	2
y_i	$\frac{1}{2}$	5	$\frac{5}{2}$	-1	$\frac{1}{2}$

Solution.

A relatively short solution is the following: Since both x_1 and x_5 have the value $\frac{1}{2}$ we can write (1 P.)

$$p(x) = q(x) + \frac{1}{2} \qquad \text{or } q(x) = p(x) - \frac{1}{2}$$

such that $q(-2) = q(2) = 0$. This means we can write (1 P.)

$$q(x) = (x-2)(x+2)r(x) = (x^2-4)r(x)$$

Plugging the remaining points into this, we obtain (3 P.)

$$\begin{aligned}
5 &= p(-1) = q(-1) + \frac{1}{2} = \left((-1)^2 - 4\right)r(-1) + \frac{1}{2} = -3r(-1) + \frac{1}{2} \\
&\Rightarrow -3r(-1) = \frac{9}{2} \Rightarrow r(-1) = -\frac{3}{2} \\
\frac{5}{2} &= p(0) = q(0) + \frac{1}{2} = \left(0^2 - 4\right)r(-1) - \frac{1}{2} = -4r(0) + \frac{1}{2} \\
&\Rightarrow -4r(0) = 2 \Rightarrow r(0) = -\frac{1}{2} \\
-1 &= p(1) = q(1) + \frac{1}{2} = \left(1^2 - 4\right)r(1) + \frac{1}{2} = -3r(1) + \frac{1}{2} \\
&\Rightarrow -3r(1) = -\frac{3}{2} \Rightarrow r(1) = \frac{1}{2}
\end{aligned}$$

And we can easily see that $r(x) = x - \frac{1}{2}$. Hence we obtain $q(x) = (x^2-4)(x-\frac{1}{2})$ and therefore

$$p(x) = q(x) + \frac{1}{2} = x^3 - \frac{1}{2}x^2 - 4x + \frac{5}{2}$$

(3 P.)

Newton Scheme.

For the newton Scheme we obtain the polynomials (3 P.)

$$
\begin{aligned}
w_0(x) &= 1\\
w_1(x) &= (x+2)\\
w_2(x) &= (x+2)(x+1) = x^2+3x+2\\
w_3(x) &= (x+2)(x+1)x = x^3+3x^2+2x\\
w_4(x) &= (x+2)(x+1)x(x-1) = x^4+3x^3+2x^2-x^3-3x^2-2x = x^4+2x^3-x^2-2x
\end{aligned}
$$

and the Newton scheme looks as follows. Note since we have equidistant nodes, the denominator is just always equal to the number of nodes involved (4 P.)

i	x_i	$y_i = f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$f[x_1, x_2, x_3, x_4, x_5]$
1	-2	$\frac{1}{2}$				
			$\frac{5-\frac{1}{2}}{-1-(-2)} = \frac{9}{2}$			
2	-1	5		$\frac{-\frac{5}{2}-\frac{9}{2}}{2} = -\frac{7}{2}$		
			$\frac{\frac{5}{2}-5}{0-(-1)} = -\frac{5}{2}$		$\frac{-\frac{1}{2}-\left(-\frac{7}{2}\right)}{3} = 1$	
3	0	$\frac{5}{2}$		$\frac{-\frac{7}{2}-\left(-\frac{5}{2}\right)}{2} = -\frac{1}{2}$		$\frac{1-1}{4} = 0$
			$\frac{-1-\frac{5}{2}}{1-0} = -\frac{7}{2}$		$\frac{\frac{5}{2}-\left(-\frac{1}{2}\right)}{3} = 1$	
4	1	-1		$\frac{\frac{3}{2}-\left(-\frac{7}{2}\right)}{2} = \frac{5}{2}$		
			$\frac{\frac{1}{2}-(-1)}{2-1} = \frac{3}{2}$			
5	2	$\frac{1}{2}$				

Hence we obtain (1 P.)

$$
\begin{aligned}
p(x) &= \frac{1}{2}w_0 + \frac{9}{2}w_1(x) - \frac{7}{2}w_2(x) + 1w_3(x) - 0w_4(x)\\
&= \frac{1}{2} + \frac{9}{2}(x+2) - \frac{7}{2}(x^2+3x+2) + (x^3+3x^2+2x)\\
&= \frac{1}{2} + \frac{9}{2}x + 9 - \frac{7}{2}x^2 - \frac{21}{2}x - 7 + x^3 + 3x^2 + 2x\\
&= x^3 - \frac{1}{2}x^2 - 4x + \frac{5}{2}\,.
\end{aligned}
$$

Lagrange Interpolation.

We can alternatively use Lagrange interpolation: $\ell_i(x) = \prod_{\substack{j=0\\ j\neq i}}^{n-1} \frac{x - x_j}{x_i - x_j}$. We obtain (5 P.)

$$\ell_0(x) = \frac{(x+1)x(x-1)(x-2)}{(-2+1)(-2)(-2-1)(-2-2)} = \frac{(x^2-1)x(x-2)}{(-1)(-2)(-3)(-4)} = \frac{1}{24}(x^4 - 2x^3 - x^2 + 2x)$$
$$\ell_1(x) = \frac{(x+2)x(x-1)(x-2)}{1(-1)(-2)(-3)} = \frac{(x^2-4)x(x-1)}{-6} = -\frac{1}{6}(x^4 - x^3 - 4x^2 + 4x)$$
$$\ell_2(x) = \frac{(x+2)(x+1)(x-2)(x-1)}{2\cdot 1(-1)(-2)} = \frac{(x^2-4)(x^2-1)}{4} = \frac{1}{4}(x^4 - 5x^2 + 4)$$
$$\ell_3(x) = \frac{(x+2)(x+1)x(x-2)}{-6} = -\frac{1}{6}(x^4 + x^3 - 4x^2 - 4x)$$
$$\ell_4(x) = \frac{(x+2)(x+1)x(x-1)}{4\cdot 3\cdot 2\cdot 1} = \frac{1}{24}(x^4 + 2x^3 - x^2 - 2x)$$

So we obtain (3 P.)

$$\begin{aligned} p(x) &= \frac{1}{2}\ell_0(x) + 5\ell_1(x) + \frac{5}{2}\ell_2(x) - \ell_3(x) + \frac{1}{2}\ell_4(x) \\ &= \Big(\frac{1}{48} - \frac{5}{6} + \frac{5}{8} + \frac{1}{6} + \frac{1}{48}\Big)x^4 + \Big(-\frac{1}{24} + \frac{5}{6} - \frac{25}{8} + \frac{1}{6} + \frac{1}{24}\Big)x^3 + \Big(-\frac{1}{48} + \frac{10}{3} - \frac{25}{8} - \frac{2}{3} - \frac{1}{48}\Big)x^2 \\ &\quad + \Big(\frac{1}{24} - \frac{10}{3} + \frac{2}{3} - \frac{1}{24}\Big)x - \frac{5}{2} \\ &= x^3 - \frac{1}{2}x^2 - 4x + \frac{5}{2} \end{aligned}$$

Problem 2. (Fixed-point and Newton iterations, 16 points)

In pipeline design for oil transport, pressure losses must be carefully estimated. They are directly proportional to a positive friction factor k, whose inverse square root $x := 1/\sqrt{k}$ is given by a non-linear equation. For a turbulent flow, the equation to find x is

$$x = g(x), \quad \text{with} \quad g(x) := -1.93\ln(x) + 15.9\,, \tag{1}$$

in which $\ln(x)$ denotes the natural logarithm, that is, the logarithm whose basis is Euler's number: $\mathrm{e} \approx 2.7183$.

a) Compute $g'(x)$.
Use the result to determine whether $g(x)$, $x > 0$, is an increasing, decreasing or a non-monotonic function.

b) Calculate the maximum and minimum values of $g(x)$ in the interval $x \in [\mathrm{e}, \mathrm{e}^3]$.

c) Show that $|g'(x)| < 1$ for $x \in [\mathrm{e}, \mathrm{e}^3]$.

d) Starting from an initial value $x^{(0)} = \mathrm{e}^2$, does the fix point iteration converge? Explain your answer.

e) For $x^{(0)} = \mathrm{e}^2$, perform the first fixed-point iteration for the solution of Equation 1.

f) For $x^{(0)} = \mathrm{e}^2$, perform the first Newton iteration for the solution of Equation 1.

Solution.

a) We differentiate $g(x)$ to get (1 P.)

$$g'(x) = -\frac{1.93}{x}\,,$$

which is always negative for $x > 0$. Therefore, $g(x)$ is a *decreasing* function.(1 P.)

b) Since $g(x)$ is decreasing, for $x \in [\mathrm{e}, \mathrm{e}^3]$ we have (1 P.)

$$g(\mathrm{e}^3) \le g(x) \le g(\mathrm{e})\,, \quad \text{that is,} \quad g([\mathrm{e}, \mathrm{e}^3]) = [10.11, 13.97] \subset (\mathrm{e}, \mathrm{e}^3)\,.$$

(1 P.)

c) Since $g'(x)$ is an increasing function, we have (1 P.)

$$g'(\mathrm{e}) \leq g'(x) \leq g'(\mathrm{e}^3), \quad \text{that is,} \quad -\frac{1.93}{\mathrm{e}} \leq g'(x) \leq -\frac{1.93}{\mathrm{e}^3}.$$

Hence, we have $|g'(x)| \leq 1.93\mathrm{e}^{-1} < 1$ for all $x \in (\mathrm{e}, \mathrm{e}^3)$. (1 P.)

Alternatively. For $x > 2$ we have $-1 < g'(x) < 0$ and hence since $2 < \mathrm{e} < \mathrm{e}^3$ the statement holds. (2 P.)

d) Since all three conditions (3 P.)

$$\begin{aligned} &(i) \quad g'(x) \in C^0[\mathrm{e}, \mathrm{e}^3], \\ &(ii) \quad g([\mathrm{e}, \mathrm{e}^3]) \subset [\mathrm{e}, \mathrm{e}^3], \\ &(iii) \quad |g'(x)| < 1 \quad \text{for all} \quad x \in [\mathrm{e}, \mathrm{e}^3] \end{aligned}$$

are fulfilled, the fixed-point iteration converges for $x^{(0)} \in [\mathrm{e}, \mathrm{e}^3]$. (1 P.)

e) The first iteration is given by (1 P.)

$$x^{(1)} = -1.93\ln(x^{(0)}) + 15.9 = 12.04.$$

f) To compute a Newton iteration, we first rewrite the non-linear equation as (2 P.)

$$f(x) = x - g(x) = x + 1.93\ln(x) - 15.9, \quad \text{so that} \quad f'(x) = 1 - g'(x) = 1 + \frac{1.93}{x}.$$

The Newton iteration then reads (2 P.)

$$x^{(1)} = x^{(0)} - \frac{f(x^{(0)})}{f'(x^{(0)})} = x^{(0)} - \frac{x^{(0)} + 1.93\ln(x^{(0)}) - 15.9}{1 + 1.93/x^{(0)}} = \frac{15.9 + 1.93\left[1 - \ln(x^{(0)})\right]}{1 + 1.93/x^{(0)}}.$$

For $x^{(0)} = \mathrm{e}^2$, we get (1 P.)

$$x^{(1)} = \frac{13.97}{1 + 1.93\mathrm{e}^{-2}} \approx 11.0768.$$

Another possible reformulation is $f(x) = g(x) - x$ then $x^{(1)} = -1.53029$ which is also a correct solution.

Problem 3. (Ralston's method, 14 points)

For the ordinary differential equation

$$y'(t) = -6y(t)\,, \quad \text{with} \quad y(0) = 1\,,$$

consider *Ralston's method* given by the following Butcher tableau:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 2/3 & 2/3 & 0 \\ \hline & 1/4 & 3/4 \end{array}$$

Using the tableau and expanding the stage derivatives k_i, we can write the solution y_{n+1} in terms of the previous one, y_n, and of the time-step size $h > 0$. More precisely:

$$y_{n+1} = R(h)\,y_n\,, \quad \text{so that} \quad y_n = [R(h)]^n\,y(0)\,,$$

in which $R(h)$ is a second-degree polynomial.

a) How many stages does this Runge–Kutta method have?

b) Determine the polynomial $R(h)$.

c) Using the expression obtained for $R(h)$, determine for what range of step sizes this algorithm is stable.

Solution.

a) The tableau shows that the method has 2 stages. (1 P.)

b) From the tableau and the ODE, we can write (6 P.)

$$k_1 = f(t_n + 0 \cdot h,\ y_n + 0 \cdot hk_1 + 0 \cdot hk_2) = f(t_n, y_n) = -6y_n\,,$$
$$k_2 = f(t_n + 2h/3,\ y_n + 2/3 \cdot hk_1 + 0 \cdot hk_2) = f(t_n + 2h/3,\ y_n - 4hy_n) = -6(y_n - 4hy_n)\,,$$
$$y_{n+1} = y_n + \frac{h}{4}(k_1 + 3k_2) = y_n - 6hy_n + 18h^2 y_n = (1 - 6h + 18h^2)y_n\,.$$

Hence, $R(h) = 1 - 6h + 18h^2$. (1 P.)

c) Stability means that y_n does *not* tend to infinity for $n \to \infty$. (1 P.)
Since $y_n = [R(h)]^n \, y(0)$ and (2 P.)

$$R(h) = 1 - 6h + 18h^2 = \frac{(6h-1)^2 + 1}{2} > 0 \quad \text{for all } h \in \mathbb{R},$$

all we need to guarantee is $R(h) \leq 1$, that is, (1 P.)

$$1 - 6h + 18h^2 \leq 1 \iff -6h + 18h^2 \leq 0 \iff 6h(3h-1) \leq 0\,.$$

Since $h > 0$ we can divide by $6h$ to obtain $3h - 1 \leq 0$, which yields $h \leq \frac{1}{3}$ (1 P.)
For any h larger than that, the solution will diverge. Hence: $h_{\max} = 1/3$, or (1 P.)

$$0 < h \leq \frac{1}{3}\,.$$

Problem 4. (Laplace transform, 12 points)

a) Compute the Laplace transform of

$$f(t) = t^2 e^{-4t}.$$

b) For a given constant $a \in \mathbb{R}$, show that the inverse Laplace transform of

$$Y(s) = \frac{s+a}{(s-2)^2}$$

is

$$y(t) = \mathcal{L}^{-1}(Y)(t) = e^{2t}[1 + (a+2)t].$$

c) Solve the initial value problem

$$y'' - 4y' + 4y = 0, \quad y'(0) = y(0) = 1,$$

using the Laplace transform.

Solution.

a) By definition we have (2 P.)

$$F(s) = \int_0^\infty t^2 e^{-4t} e^{-st} \, dt = \int_0^\infty t^2 e^{-(s+4)t} \, dt$$

Hence $F(s) = \mathcal{L}(t^2)(s+4)$ and we can look up that $\mathcal{L}(t^2) = \frac{2}{s^3}$ to obtain (2 P.)

$$F(s) = \frac{2}{(s+4)^3}.$$

Alternatively. One can also solve the integral by applying integration by parts twice. There are also several other approaches to compute this.

b) We can decompose $Y(s)$ as (2 P.)

$$Y(s) = \frac{s+a}{(s-2)^2} = \frac{s-2}{(s-2)^2} + \frac{2+a}{(s-2)^2} = \frac{1}{(s-2)} + \frac{2+a}{(s-2)^2}.$$

Using the shift theorem we obtain (2 P.)

$$y(t) = \mathcal{L}^{-1}\left[(s-2)^{-1}\right](t) + (2+a)\mathcal{L}^{-1}\left[(s-2)^{-2}\right](t) = e^{2t}[1 + (a+2)t].$$

c) Applying the Laplace transform to the ODE, with the boundary conditions we get (2 P.)

$$0 = \mathcal{L}(y'' - 4y' + 4y) = \left(s^2 Y(s) - 1 \cdot s - 1\right) - 4\left(sY(s) - 1\right) + 4Y(s)\,,$$

so that (1 P.)

$$Y(s) = \frac{s-3}{(s-2)^2}\,.$$

Now we can use what was done in b), with $a = -3$, to compute the inverse transform (1 P.)

$$y(t) = \mathcal{L}^{-1}(Y)(t) = \mathrm{e}^{2t}\left[1 + (-3 + 2)\right] = \mathrm{e}^{2t}(1 - t)\,.$$

Problem 5. (Fourier Series, 14 points)

Let g be the periodic continuation of the odd extension $f_o\colon [-\pi, \pi) \to \mathbb{R}$ for

$$f(x) = \begin{cases} 0 & \text{for } 0 \le x < \frac{\pi}{2}, \\ \frac{\pi}{2} - x & \text{for } \frac{\pi}{2} \le x \le \pi, \end{cases} \qquad x \in [0, \pi].$$

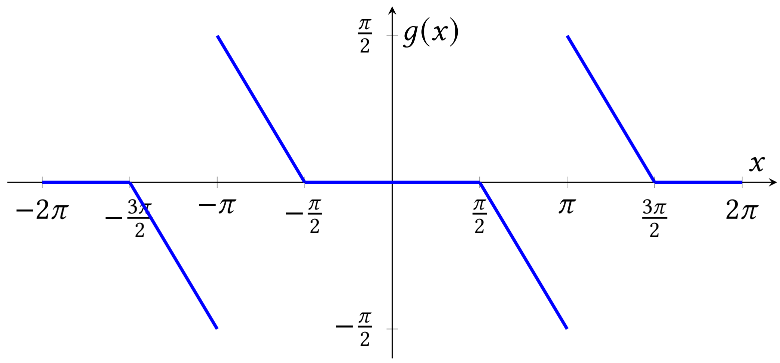
a) Sketch the function g on an interval of length of at least 2 periods.

b) Compute the real Fourier series of g. Simplify the result.

c) We denote the Fourier partial sum of the Fourier series from b) by S_n.
Let $x_0 = \pi$ and $x_1 = -\frac{\pi}{2}$. What values do the Fourier partial sums converge to, i. e. what are the limits $\lim\limits_{n\to\infty} S_n(x_0)$ and $\lim\limits_{n\to\infty} S_n(x_1)$?

Solution.

a) The sketch looks for example like (2 P.)



where the important points are at $\frac{k\pi}{2}, k = -4, \ldots, 4$.

b) The Fourier coefficients are $a_0 = a_n = 0$ for $n \in \mathbb{N}$, since g is odd. (1 P.)

For the b_n we can use the formula for odd functions and compute (checking for example the $2L$ periodic formula, then $L = \pi$) (2 P.)

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\mathrm{d}x \\ &= \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} \big(\frac{\pi}{2} - x\big)\sin(nx)\mathrm{d}x \\ &= \frac{2}{\pi}\frac{\pi}{2}\int_{\frac{\pi}{2}}^{\pi}\sin(nx)\mathrm{d}x - \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} x\sin(nx)\mathrm{d}x \end{aligned}$$

The first term we can just integrate and for the second we use integration by parts (note that integrating the sin here introduces a minus) (4 P.)

$$\begin{aligned}
b_n &= \Big[-\frac{1}{n}\cos(nx)\Big]_{\frac{\pi}{2}}^{\pi} - \frac{2}{\pi}\Big[-\frac{x}{n}\cos(nx)\Big]_{\frac{\pi}{2}}^{\pi} + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} 1\cdot\big(-\frac{1}{n}\cos(nx)\big)\mathrm{d}x\\
&= -\frac{1}{n}\Big(\cos(n\pi) - \cos\frac{n\pi}{2}\Big) + \frac{2}{n\pi}\Big(\pi\cos(n\pi) - \frac{\pi}{2n}\cos\frac{n\pi}{2}\Big) + \frac{2}{\pi}\Big[-\frac{1}{n^2}\sin(nx)\Big]_{\frac{\pi}{2}}^{\pi}\\
&= -\frac{1}{n}\cos(n\pi) + \frac{1}{n}\cos\big(\frac{n\pi}{2}\big) + \frac{2}{n}\cos(n\pi) - \frac{1}{n}\cos\big(\frac{n\pi}{2}\big) - \frac{2}{\pi n^2}\Big(\sin(n\pi) - \sin\frac{n\pi}{2}\Big)\\
&= \frac{1}{n}\cos(n\pi) + \frac{2}{\pi n^2}\sin\frac{n\pi}{2}
\end{aligned}$$

Now $\cos(n\pi) = (-1)^n$ and $\sin\frac{n\pi}{2} = 0$ if n is even and $\sin\frac{(2k+1)\pi}{2} = (-1)^k$ so we can simplify further (1 P.)

$$g \sim \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(nx) + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\big((2k+1)x\big)$$

c) The function g is piecewise continuously differentiable and the limits of both the function g as well as the derivative g' exist at every point. The left and right limits of g at x_0 and x_1 are They are (2 P.)

$$\lim_{x\to x_0^-} g(x) = -\frac{\pi}{2}, \quad \lim_{x\to x_0^+} g(x) = \frac{\pi}{2}, \quad \lim_{x\to x_1^-} g(x) = \lim_{x\to x_1^+} g(x) = 0$$

since at x_1 the function g is even continuous. Hence the Fourier partial sum converges to $\lim_{n\to\infty} S_n(x_0) = \frac{1}{2}\Big(\frac{\pi}{2} - \frac{\pi}{2}\Big) = 0$ and $\lim_{n\to\infty} S_n(x_1) = 0$. (2 P.)

The second limit at x_0 can *alternatively* also be obtained, seeing that setting $x = x:0$ in the series from the previous point, all terms vanish.

Problem 6. (Fourier Transform, 8 points)

Let $\lambda, L > 0$ be given. Compute the Fourier Transform of $f(x) = \begin{cases} \cos(\lambda x), & \text{for } |x| \leq L, \\ 0 & \text{else.} \end{cases}$

Solution.

We can use Eulers identity on the cosine to obtain (2 P.)

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} \cos(\lambda x) \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi}} \int_{-L}^{L} \frac{1}{2}\Big(\mathrm{e}^{\mathrm{i}\lambda x} + \mathrm{e}^{-\mathrm{i}\lambda x}\Big) \mathrm{e}^{-\mathrm{i}\omega x} \, \mathrm{d}x \\ &= \frac{1}{2\sqrt{2\pi}} \int_{-L}^{L} \mathrm{e}^{\mathrm{i}x(\lambda-\omega)} + \mathrm{e}^{-\mathrm{i}x(\lambda+\omega)} \, \mathrm{d}x \end{aligned}$$

For $\lambda \neq \pm\omega$ we can compute the antiderivative of both terms (3 P.)

$$\begin{aligned} \hat{f}(\omega) &= \frac{1}{2\sqrt{2\pi}} \Big(\frac{1}{\mathrm{i}(\lambda-\omega)} \mathrm{e}^{\mathrm{i}x(\lambda-\omega)} - \frac{1}{\mathrm{i}(\lambda+\omega)} \mathrm{e}^{-\mathrm{i}x(\lambda+\omega)} \Big|_{-L}^{L} \Big) \\ &= \frac{1}{\sqrt{2\pi}} \Big(\frac{1}{2\mathrm{i}(\lambda-\omega)} \big(\mathrm{e}^{\mathrm{i}L(\lambda-\omega)} - \mathrm{e}^{-\mathrm{i}L(\lambda-\omega)}\big) + \frac{1}{2\mathrm{i}(\lambda+\omega)} \big(\mathrm{e}^{\mathrm{i}L(\lambda+\omega)} - \mathrm{e}^{-\mathrm{i}L(\lambda+\omega)}\big) \Big) \end{aligned}$$

where in the last line the second summand has a + upfront since we switched the order of the terms in the inner difference (note that for $x = -L$ the minus sign vanishes but this is the first term). Now both summands, especially already with the 2i in the denominator look like sine functions, so we can rewrite this to (2 P.)

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \Big(\frac{\sin(L(\lambda-\omega))}{\lambda-\omega} + \frac{\sin(L(\lambda+\omega))}{\lambda+\omega} \Big)$$

Extending both fractions by L and using the definition of $\operatorname{sinc}(x) = \frac{\sin x}{x}$ we can simplify this to

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \Big(L \operatorname{sinc}(L(\lambda-\omega)) + \operatorname{sinc}(L(\lambda+\omega)) \Big)$$

Since $\operatorname{sinc}(0) = 1$ we also obtain the same solution for $\lambda = \pm\omega$ where either the first or the second integral integrates to $2L$. (1 P.)

Alternatively.
One can use that the cosine is odd and one of the trigonometric identities for products of cosines (4 P.)

$$\begin{aligned}
\hat{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-L}^{L}\cos(\lambda x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x = \frac{1}{\sqrt{2\pi}}\int_{-L}^{0}\cos(\lambda x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x + \frac{1}{\sqrt{2\pi}}\int_{0}^{L}\cos(\lambda x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \\
&= \frac{2}{\sqrt{2\pi}}\int_{0}^{L}\cos(\lambda x)\big(\mathrm{e}^{\mathrm{i}\omega x} + \mathrm{e}^{-\mathrm{i}\omega x}\big)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{L}\cos(\lambda x)\cos(\omega x)\,\mathrm{d}x \\
&= \frac{1}{\sqrt{2\pi}}\int_{0}^{L}\cos\big((\lambda-\omega)x\big) + \cos\big((\lambda+\omega)x\big)\,\mathrm{d}x
\end{aligned}$$

Where again for $\lambda \neq \pm\omega$ we can easily determine the stem functions (2 P.)

$$\hat{f}(\omega) = \frac{1}{\sqrt{2\pi}}\left(\frac{\sin\big((\lambda-\omega)x\big)}{\lambda-\omega} + \frac{\sin\big((\lambda+\omega)x\big)}{\lambda+\omega}\Bigg|_0^L\right)$$

which is the same as in the first approach. (2 P.)

Problem 7. (Discrete Fourier Transform, 8 points)

For the signal $\mathbf{f} = \left(\frac{1}{2}, 1, \frac{1}{2}, 0\right) \in \mathbb{R}^4$ we want to consider the Discrete Fourier Transform $\hat{\mathbf{f}} = \mathcal{F}_4\mathbf{f}$.

a) What does the matrix $\mathcal{F}_4$ look like?

b) Compute $\hat{\mathbf{f}}$.

c) Let $c \in \mathbb{R}$ be given and assume that for another signal $\mathbf{g}$ we obtain $\hat{\mathbf{g}} = \left(\hat{g}_0, \hat{g}_1, \hat{g}_2, \hat{g}_3\right) = \mathcal{F}_4\mathbf{g}$ with $\hat{g}_1 = \hat{g}_3 = c$ and $\hat{g}_0 = \hat{g}_2 = 0$. What is the simplest function $g(x)$ that could have been sampled?

   Hint. Think of a bandlimited function or a trigonometric polynomial $g(x)$.

d) Is the inverse Fourier transform $\mathbf{h} = \mathcal{F}_8^{-1}\hat{\mathbf{h}}$ of $\hat{\mathbf{h}} = \left(0, 0, 0, 0, 0, 0, 1, 0\right)$ real-valued?

Solution.

a) By definition we have $\mathcal{F}_N = \left(\mathrm{e}^{-2\pi \mathrm{i} jk/N}\right)_{j,k=0}^{N-1}$. (1 P.)

   In this problem this leads to the Fourier matrix (1 P.)

$$\mathcal{F}_4 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{pmatrix}$$

   Depending on which definition is used, a factor $\frac{1}{4}$ or $\frac{1}{\sqrt{4}} = \frac{1}{2}$ is also possible

b) We obtain (2 P.)

$$\hat{\mathbf{f}} = \mathcal{F}_4\mathbf{f} = \mathcal{F}_4 \begin{pmatrix} \hat{f}_0 \\ \hat{f}_1 \\ \hat{f}_2 \\ \hat{f}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} + 1 + \frac{1}{2} + 0 \\ \frac{1}{2} - \mathrm{i} - \frac{1}{2} + 0\mathrm{i} \\ \frac{1}{2} - 1 + \frac{1}{2} - 0 \\ \frac{1}{2} + \mathrm{i} - \frac{1}{2} - 0\mathrm{i} \end{pmatrix} = \begin{pmatrix} 2 \\ -\mathrm{i} \\ 0 \\ \mathrm{i} \end{pmatrix}$$

   or the two scaled versions mentioned at the end of a).

c) If we obtain the Fourier transform $\hat{\mathbf{g}} = \big(0, c, 0, c\big)$ then the inverse Fourier transform is $\mathbf{g} = \frac{1}{2}\big(c, 0, -c, 0\big)$. Which is a cosine. (2 P.)

Alternatively one can argue that with a Fourier shift these correspond to $c_0(g) = 0 = c_{-2}(g) = 0$ and $c_{\pm 1}(g) = c$ and this corresponds to $a_1 = 2c_1 = 2c$ and all other coefficients $a_0 a_n = 0$ $(n \neq 1)$ and $b_n = 0$, $n = 1, 2, \ldots$. Hence $\mathbf{g}$ are the sampling values $g_j = g(t_j)$ at $t_j = \frac{\pi j}{2}$, $j = 0, 1, 2, 3$ of

$$g(x) = 2c\cos(x).$$

d) No, since with the Fourier shift this corresponds to $c_{-4} = \hat{h}_4$, $c_{-3} = \hat{h}_5$, ..., so with $c_{-1} = \hat{h}_7 = 1$ but $c_1 = \hat{h}_1 = 0$ the condition that $c_k = \overline{c_{-k}}$ does not hold. (2 P.)

Alternatively one can argue that the inverse Fourier transform consists of the seventh column of $\mathcal{F}_8^{-1}$ and that this column contains for example i which is then an entry of $\mathbf{h}$.

Problem 8. (Heat equation, 12 points)

Consider the following partial differential equation: find $u(x,t)$ that fulfils

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \sin(3\pi x)\,, \tag{2}$$

$$u(0,t) = u(1,t) = 0\,, \quad t \geq 0 \tag{3}$$

$$u(x,0) = 0\,, \quad 0 \leq x \leq 1\,. \tag{4}$$

which is an inhomogeneous heat equation.

a) Consider the functions $u_n(x,t) = \mathrm{e}^{-\omega_n^2 t}\sin(\omega_n x)$, with $n \in \mathbb{N}$ and $\omega_n \in \mathbb{R}$. Determine the values ω_n such that $u_n(0,t) = u_n(1,t) = 0$, and show that

$$\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial x^2} = 0\,.$$

b) Show that $v(x,t) = \dfrac{1}{9\pi^2}\sin(3\pi x)$ satisfies the equation

$$\frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = \sin(3\pi x)$$

and the boundary conditions $v(0,t) = v(1,t) = 0$.

c) Using superposition, the general solution for Eq. (2) with boundary conditions (3) can be written as

$$u(x,t) = \frac{1}{9\pi^2}\sin(3\pi x) + \sum_{n=1}^{\infty} B_n \mathrm{e}^{-{\omega_n}^2 t}\sin(\omega_n x)\,.$$

Determine the real coefficients B_n so that the initial condition (4) is satisfied.

Solution.

a) We have (2 P.)

$$\frac{\partial u_n}{\partial t} - \frac{\partial^2 u_n}{\partial x^2} = -{\omega_n}^2 \mathrm{e}^{-{\omega_n}^2 t}\sin(\omega_n x) - \omega_n(-\omega_n)\mathrm{e}^{-{\omega_n}^2 t}\sin(\omega_n x) = 0 \quad \text{for all } \ \omega_n \in \mathbb{R}\,.$$

The boundary condition at $x = 0$ is already satisfied for all $\omega_n \in \mathbb{R}$ (1 P.)

The condition at $x = 1$ require $\sin(\omega_n \cdot 1) = 0$, since $\mathrm{e}^{-\omega_n^2 t} \neq 0$, that is, (1 P.)

$$\omega_n = n\pi\,, \quad \text{with} \quad n \in \mathbb{N}\,.$$

We can thus write $u_n(x,t) = \mathrm{e}^{-(n\pi)^2 t}\sin(n\pi x)$. (1 P.)

b) We can verify by computing the derivatives (3 P.)

$$\frac{\partial v}{\partial t} = 0 \quad \text{and} \quad \frac{\partial^2 v}{\partial x^2} = -\sin(3\pi x), \quad \text{that} \quad \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = 0 - (-\sin(3\pi x)) = \sin(3\pi x).$$

At the boundary we obtain (2 P.)

$$v(0,t) = \frac{1}{9\pi^2}\sin(0) = 0 \quad \text{and} \quad v(1,t) = \frac{1}{9\pi^2}\sin(3\pi) = 0.$$

c) The initial conditions require (1 P.)

$$0 = u(x,0) = \frac{1}{9\pi^2}\sin(3\pi x) + \sum_{n=1}^{\infty} B_n \sin(n\pi x), \quad \text{that is,} \quad \sum_{n=1}^{\infty} B_n \sin n\pi x = -\frac{1}{9\pi^2}\sin(3\pi x).$$

Hence, B_n are the coefficients of a Fourier series. Since the right-hand side is one of the summands (with a prefactor), we can directly read of the coefficients(1 P.)

$$B_n = \begin{cases} -\frac{1}{9\pi^2} & \text{if } n = 3 \\ 0 & \text{else.} \end{cases}$$

Problem 9. (Separation of Variables, 8 points)

Consider the fourth order PDE

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^4 u}{\partial x^4} = 0, \qquad x \in [0, 1], t \geq 0.$$

a) Use the Ansatz $u(x, t) = F(x)G(t)$ to transform the PDE into a system of ODEs.

b) Verify that for $\beta > 0, A, B \in \mathbb{R}$ all functions of the form

$$F(x) = A\sin(\beta x) + B\cos(\beta x) \tag{5}$$

satisfy the ODE for F in a).

We add the following (four) boundary conditions for $t > 0$

$$\begin{cases} u(0, t) = u(1, t) = 0, \\ \dfrac{\partial^2}{\partial x^2}u(0, t) = \dfrac{\partial^2}{\partial x^2}u(1, t) = 0. \end{cases}$$

For which β, A, B do the functions in (5) fulfil the boundary conditions?

c) How many initial conditions would you expect to be given to have a unique solution for G?

Hint. Compare this PDE to what you know about the heat and wave equation and their second ODE to solve.

d) Which other non-trivial function(s) F also fulfil the ODE derived in a)? State an example. You can ignore the boundary conditions.

Solution.

a) We use the Ansatz $u(x, t) = F(x)G(t)$ to obtain

$$\frac{\partial^2}{\partial t^2}u(x, t) = F(x)G''(t) = -F^{(4)}(x)G(t) = -\frac{\partial^4}{\partial x^4}u(x, t)$$

which we rearrange to (1 P.)

$$\frac{G''(t)}{G(t)} = -\frac{F^{(4)}(x)}{F(x)} = -k$$

where k is a constant. We obtain the two ODEs (1 P.)

$$F^{(4)}(x) - kF(x) = 0$$
$$G''(t) + kG(t) = 0$$

b) In the Ansatz from a) the boundary conditions yield that a solution to the ODE w.r.t. F has to fulfil

$$u(0,t) = F(0)G(t) = F(1)G(t) = u(1,t) = 0$$

for all $t > 0$. Hence $F(0) = F(1) = 0$.
Similarly the other boundary conditions yield that $F''(0) = F''(1) = 0$. (1 P.)

We first verify that the given form fulfils the ODE. We have

$$\begin{aligned} F'(x) &= A\beta\cos(\beta x) - B\beta\sin(\beta x) \\ F''(x) &= -A\beta^2\sin(\beta x) - B\beta^2\cos(\beta x) \\ F^{(3)}(x) &= -A\beta^3\cos(\beta x) + B\beta^3\sin(\beta x) \\ F^{(4)}(x) &= A\beta^4\cos(\beta x) + B\beta^4\cos(\beta x) = \beta^4 F(x) \end{aligned}$$

Hence it fulfils the ODE for F. (1 P.)

For the boundary conditions, we obtain

$$0 = F(0) = A \cdot 0 + B \cdot 1 = 0 \Rightarrow B = 0$$

So we can continue with just functions of the form $F(x) = A\sin(\beta x)$. From the second boundary condition we obtain (1 P.)

$$0 = F(1) = A\sin(\beta) \Rightarrow \beta = n\pi.$$

The third and fourth boundary conditions read (1 P.)

$$0 = F''(0) = -A\beta^2\sin(0) \qquad \text{and} \qquad 0 = F(1) = -A\beta^2\sin(\beta)$$

The first is true for any β, the second holds for the already found $\beta = n\pi$, $\beta \in \mathbb{N}$.

c) We need two initial conditions. (1 P.)

An example would be ok as well, we usually need something like

$$\begin{cases} u(x,0) = 0 \\ \dfrac{\partial}{\partial t}u(x,0) = 0 \end{cases}$$

to get a unique solution (for each $\beta = n\pi$, $n \in \mathbb{N}$ or in other words $k = (n\pi)^4$ from b)) in the ODE for G.

d) For example $Ce^{\pm\beta x}$, $\beta > 0$ fulfils the ODE with $k = \beta^4$ as well, or phrased differently, $D\sinh(\beta x)$ and $E\cosh(\beta x)$ do. (1 P.)

Formula Sheet.

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Fourier Transform. The Fourier Transform $\hat{f} = \mathcal{F}(f)$ and its inverse $f = \mathcal{F}^{-1}(\hat{f})$ are

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \quad \text{and} \quad f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega$$

Laplace Transform. The Laplace transform $F(s)$ of $f(t)$, $t \geq 0$, reads

$$F(s) = \int_0^{\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

List of Fourier Transforms.

$f(x)$	$\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}\mathrm{e}^{-\frac{\omega^2}{4a}}$
$\mathrm{e}^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{a}{\omega^2+a^2}$
$\frac{1}{x^2+a^2}$ for $a > 0$	$\sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-a\lvert\omega\rvert}}{a}$
$\begin{cases} 1 & \text{for } \lvert x\rvert < a \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega a)}{\omega}$

List of Laplace Transforms.

$f(t)$	$F(s)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, see Note[(a)]
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$\mathrm{e}^{-sa}F(s)$
$\delta(t-a)$	e^{-sa}

[(a)] where for $n \in \mathbb{N}$ we have $\Gamma(n+1) = n!$

Trigonometric identities.

- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha-\beta) + \sin(\alpha+\beta)\big)$
- $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$
- $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

We also discussed the sinus cardinalis $\operatorname{sinc}(x) = \frac{\sin x}{x}$.

Fourier Series. For a 2π-periodic function f we can write

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{ikx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

with coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \quad n = 0,1,2,\ldots, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x, \quad n = 1,2,\ldots,$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-ikx}\,\mathrm{d}x, \quad k \in \mathbb{Z}.$$