NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4135 Matematikk 4D**

Solution

Academic contact during examination:

Phone:

Examination date: August 18, 2022

Examination time (from–to): 09:00–13:00

Permitted examination support material: C.
One sheet A4 paper, approved by the department (yellow sheet, "gul ark") with own handwritten notes.
Certain simple calculators.

Other information:

- All answers have to be justified, and they should include enough details in order to see how they have been obtained.
- Good Luck! | Lykke til! | Viel Glück!

Language: English

Number of pages: 21

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	2-sidig ☒
sort/hvit ☒	farger ☐
skal ha flervalgskjema ☐	

Date Signature

In the exam one could obtain 100 points and the exam was graded using the usual grading scheme, i.e.

A	B	C	D	E	F
100–89	88–77	76–65	64–53	52–41	40 and less

Problem 1. (Interpolation, 12 points)

Consider the data points

x_i	-2	0	1
$f(x_i)$	2	2	4

a) Use Lagrange interpolation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$.

b) Determine the Newton form of the interpolating polynomial and express the resulting polynomial in the form $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$.

c) Now add the data point $(x_3, f_3) = (2, 6)$ and compute the resulting interpolation polynomial for the given 4 data points.

Solution.

a) Lagrange polynomial and resulting interpolation polynomial for the first 3 data points:

$$\begin{aligned}
L_0 &= \frac{x\,(x-1)}{6} \\
L_1 &= -\frac{(x-1)\,(x+2)}{2} \\
L_2 &= \frac{x\,(x+2)}{3} \\
p_2(x) &= \frac{x\,(x-1)}{3} - (x-1)\,(x+2) + \frac{4x\,(x+2)}{3} = \frac{2x^2}{3} + \frac{4x}{3} + 2
\end{aligned}$$

b) The Newton polynomials for the first three data points are

$$\begin{aligned}
\omega_0 &= 1 \\
\omega_1 &= x + 2 \\
\omega_2 &= x\,(x+2)
\end{aligned}$$

The divided difference table is given by

x_i	$f(x_i)$		
-2	2		
0	2	0	
1	4	2	$2/3$

and thus the interpolation polynomial in Newton form is

$$p_2(x) = 2 \cdot 1 + 0 \cdot (x+2) + 2/3 \cdot x\,(x+2) = \frac{2x^2}{3} + \frac{4x}{3} + 2$$

c) We compute the interpolation polynomial in Newton form which only requires to extend the divided difference table from a) accordingly using the 4th data point.

x_i	$f(x_i)$			
-2	2			
0	2	0		
1	4	2	$2/3$	
2	6	2	0	$-1/6$

The 4th Newton polynomial and final interpolation polynomial are given by

$$\begin{aligned} \omega_3 &= x\,(x-1)\,(x+2) \\ p_3(x) &= 2 \cdot 1 + 0 \cdot (x+2) + 2/3 \cdot x\,(x+2) - 1/6 \cdot x\,(x-1)\,(x+2) \\ &= -\frac{x^3}{6} + \frac{x^2}{2} + \frac{5x}{3} + 2 \end{aligned}$$

Problem 2. (Quadrature, 8 points)

a) Given are the quadrature points $x_0 = -2$, $x_1 = 0$ and $x_2 = 1$. Determine corresponding weights ω_0, ω_1 and ω_2 such that the quadrature rule $Q[f](-2,1) := \sum_{i=0}^{2} \omega_i f(x_i)$ has *at least* degree of exactness 2 on the interval $[-2, 1]$.

   Hint: You might want to solve Problem 1 first to save you some time.

b) Imagine you have a composite quadrature rule $CQ[\cdot; h]$. Here h denotes the length of the subintervals used the composite quadrature rule. Now you perform a convergence study using the function $f(x) = \cos(x) + \sin(x)$ on the interval $[0, 1]$ and you obtain the following table

h	0.5	0.25	0.125	0.0625
$E(h)$	0.8192	0.0513	0.0032	0.000201

   where $E(h) = \int_0^1 f(x)dx - CQ[f; h]$ is the quadrature error as a function of h. What convergence rate do expect for the composite quadrature rule to have and why?

Solution.

a) We need to compute the Lagrange polynomials L_0, L_1 and L_2 associated with the quadrature points x_0, x_1, x_2. Then ω_i are determined by

$$\omega_i = \int_{-2}^{1} L_i(x)dx$$

   Note that we had the same points in the Problem 1, so we do not need to recompute L_i, we only need to integrate the computed Lagrange polynomials

$$\omega_0 = \int_{-2}^{1} \frac{x\,(x-1)}{6} dx = 3/4$$
$$\omega_1 = -\int_{-2}^{1} \frac{(x-1)\,(x+2)}{2} dx = 9/4$$
$$\omega_2 = \int_{-2}^{1} \frac{x\,(x+2)}{3} dx = 0$$

b) For each bisection we observe that the error is reduced by a factor of $16 = 2^4$, thus the convergence order seems to be 4.

Problem 3. (Partial derivatives, 10 points)

Consider the function $f : \mathbb{R}^3 \to \mathbb{R}$

$$f(x, y, z) = x^2 \sin(y)\mathrm{e}^{-z} - 2x^2$$

For this function, compute each of the following partial derivatives:

$$\frac{\partial f}{\partial x}, \quad \frac{\partial^2 f}{\partial y^2} \quad \text{and} \quad \frac{\partial^3 f}{\partial x \partial y \partial z}.$$

Solution.

The derivatives are (3+3+4 P.)

$$\begin{aligned} \frac{\partial f}{\partial x} &= 2x \sin(y)\mathrm{e}^{-z} - 2x \\ \frac{\partial^2 f}{\partial y^2} &= -x^2 \sin(y)\mathrm{e}^{-z} \\ \frac{\partial^3 f}{\partial x \partial y \partial z} &= -2x \cos(y)\mathrm{e}^{-z} \end{aligned}$$

Problem 4. (Nonlinear Equations (Fixed Point Theory), 12 pts)

Let the sequence $(x_n)_{n\in\mathbb{N}_0}$ be defined by

$$x_{n+1} := \frac{3x_n + 1}{2x_n + 1}$$

with the starting value $x_0 := 1$.

a) Check whether the sequence $(x_n)_{n\in\mathbb{N}_0}$ is convergent and, if so, compute the limit value $\widehat{x} := \lim\limits_{n\to\infty} x_n$.

b) Determine an upper bound for $|x_5 - \widehat{x}|$ without calculating x_5.

Solution.

a) The elements of the sequence lie obviously all in $\mathbb{R}_0^+$. The given sequence is an iteration sequence to the fixed point equation

$$x = \underbrace{\frac{3x+1}{2x+1}}_{f(x)}, \quad x \in \mathbb{R}_0^+.$$

Here we can directly compute the solutions of $x = f(x)$, which must be in $\mathbb{R}_0^+$:

$$x = \frac{3x+1}{2x+1} \quad \Leftrightarrow \quad 2x^2 + x = 3x + 1 \quad \Leftrightarrow \quad \boxed{2x^2 - 2x - 1 = 0}$$

$$x = \frac{2 \pm \sqrt{4 + 2\cdot 2\cdot 1}}{2\cdot 2} = \begin{cases} \frac{1}{2}(1+\sqrt{3}) \in \mathbb{R}_0^+ \\ \frac{1}{2}(1-\sqrt{3}) \notin \mathbb{R}_0^+ \end{cases}$$

If the sequence converges, we have:

$$\lim_{n\to+\infty} x_n = \frac{1}{2}(1+\sqrt{3}) =: \hat{x}.$$

With $x \in \mathbb{R}_0^+$, it holds $f(x) \in \mathbb{R}_0^+$.
Therefore: $f(\mathbb{R}_0^+) \subseteq \mathbb{R}_0^+$.
<u>However:</u> In $\mathbb{R}_0^+$, we have:

$$f'(x) = \frac{3(2x+1) - (3x+1)2}{(2x+1)^2} = \frac{1}{(2x+1)^2} > 0$$

Therefore: $f'(x) = 1$ for $x = 0$. Therefore, we consider f only on $D := [1, +\infty[$. It holds $f(D) \subseteq D$, as $f(x) > 1$ for all $x > 0$, and $f'(x) \leq \left(\frac{1}{3}\right)^2$ for $x \in D$. Thus, we can use the Banach-fixed point theorem: $\hat{x}$ is a limit point of the given sequence.

b) $|x_5 - \hat{x}| \leq \dfrac{|x_1 - x_0|}{1 - \frac{1}{9}} \left(\dfrac{1}{9}\right)^5 = \dfrac{3}{8} \left(\dfrac{1}{9}\right)^5 \approx 6.35066 \cdot 10^{-6}$.

Problem 5. (Laplace transform, 12 points)

a) Find $y(t)$, $t \geq 0$ such that $y(0) = 0$ and

$$\int_0^t y'(t-u)y(u)\,\mathrm{d}u = \frac{1}{6}t^4.$$

b) Let $a > 0$ be given. Compute the inverse Laplace transform of

$$F(s) = \frac{-3a - 2(s+1)}{(s+1)(s-2)}$$

c) Solve the initial value problem

$$\begin{aligned} y'' - y' - 2y &= 0 \\ y(0) &= -2 \\ y'(0) &= -1 \end{aligned}$$

using the Laplace transform.

Hint: You might want to solve b) first.

Solution.

a) We set $Y(s) = \mathcal{L}(y(t))$ and obtain that (1 P.)

$$\mathcal{L}(y'(t)) = sY(s) - y(0) = sY(s)$$

Taking the Laplace transform of the given equation, we can use that the left hand side is a convolution. We obtain

$$sY(s)Y(s) = \frac{1}{6}\mathcal{L}(t^4) = \frac{1}{6}\frac{24}{s^5} = \frac{4}{s^5}$$

We can divide by s to obtain (1 P.)

$$(Y(s))^2 = \frac{4}{s^6}$$

So we obtain (1 P.)

$$Y(s) = \sqrt{\frac{4}{s^6}} = \pm\frac{2}{s^3}$$

Taking the inverse Laplace transform we obtain (1 P.)

$$y(t) = \pm t^2$$

b) We perform a partial fraction decomposition (2 P.)

$$F(s) = \frac{-3a - 2(s+1)}{(s+1)(s-2)} = \frac{-3a - 2 - 2s}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$
$$= \frac{A(s-2) + B(s+1)}{(s+1)(s-2)} = \frac{s(A+B) + (B-2A)}{(s+1)(s-2)}$$

So we obtain $B - 2A = -3a - 2$ and $A + B = -2$ and hence we get $A = a$ and $B = -A - 2 = a - 2$. (1 P.)

We obtain from $F(s) = \frac{a}{s+1} - \frac{2+a}{s-2}$ that (2 P.)

$$f(t) = \mathcal{L}^{-1}(F) = a\mathrm{e}^{-t} - (2+a)\mathrm{e}^{2t}$$

c) We apply the Laplace transform to the ODE to obtain (2 P.)

$$s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) - 2Y(s) = s^2Y(s) + 2s + 1 - sY(s) - 2 - 2Y(s)$$
$$= (s^2 - s - 2)Y(s) + 2s - 1 = 0$$

and hence (1 P.)

$$Y(s) = \frac{1 - 2s}{s^2 - s - 2} = \frac{3 - 2(s+1)}{(s+1)(s-2)}$$

which is the same as in the previous subproblem with $a = -1$, so we obtain (1 P.)

$$y(t) = -\mathrm{e}^{-t} - \mathrm{e}^{2t}.$$

Problem 6. (Fourier Series, 14 points)

Let

$$f(x) = \begin{cases} \frac{\pi}{2} - x & \text{for } 0 \le x < \frac{\pi}{2}, \\ 0 & \text{for } \frac{\pi}{2} \le x \le \pi, \end{cases} \qquad x \in [0, \pi].$$

We consider the odd extension f_o, the even extension f_e as functions on $[-\pi, \pi)$. We define g to be the periodic continuation of the even extension f_e.

a) Sketch the function f_o. In the same plot, also sketch g on an interval of length of at least 2 periods.

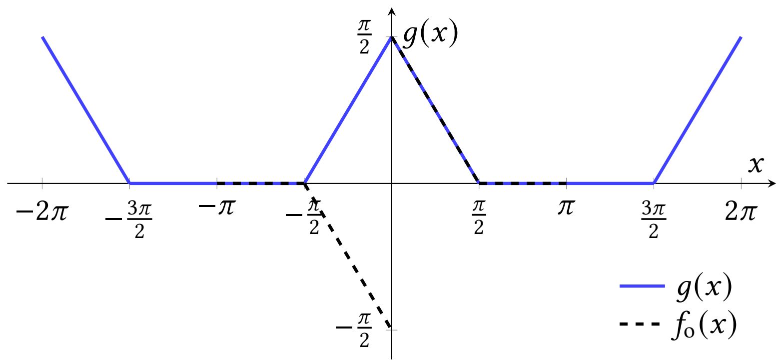
b) Compute the real Fourier series of g.

c) We denote the Fourier partial sum of the Fourier series from b) by S_n.
Let $x \in \mathbb{R}$ be given. What value does the Fourier partial sum converge to, i. e. what is the value of $\lim_{n\to\infty} S_n(x)$?

d) We denote by $S_n'(x)$ the derivative of the Fourier series from b), which is again a Fourier series. Compute $\lim_{n\to\infty} S_n'(0)$.
Hint. It might be helpful to first think about what $S_n'(x)$ converges to as a function.

Solution.

a) The sketch looks for example like (3 P.)



where the important points are at $\frac{k\pi}{2}$, $k = -4, \dots, 4$.

b) The Fourier coefficients are $b_n = 0$ for $n \in \mathbb{N}$, since g is odd. (1 P.)

For the a_0, a_n we can use the formula for even functions and compute (checking for example the $2L$ periodic formula, then $L = \pi$).

We obtain (1 P.)

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{\pi}{2} - x\,\mathrm{d}x = \frac{2}{\pi}\Big[\frac{\pi}{2}x - \frac{1}{2}x^2\Big]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\Big(\frac{\pi^2}{4} - \frac{\pi^2}{8}\Big) = \frac{\pi}{4}.$$

For the remaining terms we compute (2 P.)

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \big(\frac{\pi}{2} - x\big)\cos(nx)\,\mathrm{d}x \\ &= \frac{2}{\pi}\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \cos(nx)\,\mathrm{d}x - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} x\cos(nx)\,\mathrm{d}x \end{aligned}$$

The first term we can just integrate and for the second we use integration by parts (note that integrating the cos here does not introduce a minus) (3 P.)

$$\begin{aligned} a_n &= \Big[\frac{1}{n}\sin(nx)\Big]_0^{\frac{\pi}{2}} - \frac{2}{\pi}\Big[\frac{x}{n}\sin(nx)\Big]_0^{\frac{\pi}{2}} + \frac{2}{\pi}\int_0^{\frac{\pi}{2}} 1\cdot\frac{1}{n}\sin(nx)\,\mathrm{d}x \\ &= \frac{1}{n}\Big(\sin\big(\frac{n\pi}{2}\big) - \sin(0)\Big) - \frac{2}{\pi}\Big(\frac{\pi}{2n}\sin\big(\frac{n\pi}{2}\big) - 0\sin(0)\Big) + \frac{2}{\pi}\Big[-\frac{1}{n^2}\cos(nx)\Big]_0^{\frac{\pi}{2}} \\ &= \frac{1}{n}\sin\big(\frac{n\pi}{2}\big) + \frac{1}{n}\sin\big(\frac{n\pi}{2}\big) - \frac{2}{\pi n^2}\Big(\cos\frac{n\pi}{2} - 1\Big) \\ &= \frac{2}{n}\sin\big(\frac{n\pi}{2}\big) + \frac{2}{\pi n^2}\Big(\cos\frac{n\pi}{2} - 1\Big) \end{aligned}$$

c) The function g is piecewise continuously differentiable and hence for every x the Fourier partial sum converges to g. (2 P.)

d) The derivative of the Fourier partial sum is itself a Fourier series.

Indeed it approximates $g'(x) = \begin{cases} 1 & \text{for } -\frac{\pi}{2} < x < 0 \\ -1 & \text{for } 0 < x < \frac{\pi}{2} \\ 0 & \text{else} \end{cases}$ $, x \in [-\pi,\pi)\setminus\{-\frac{\pi}{2}, 0, \frac{\pi}{2}\}$,

which is not defined for $x \in \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\}$ since g is not differentiable there. The Fourier series however converges to the mean of the limits, so we get $\lim_{n\to\infty} S_n'(0) = 0$. (2 P.)

Problem 7. (Fourier Transform, 8 points)

Use the Fourier transform of $f(x) = \mathrm{e}^{-ax^2}$, $a > 0$, to compute a closed form of

$$h(x) = \mathrm{e}^{-2x^2} * \mathrm{e}^{-2x^2}.$$

Solution.

We can use the convolution theorem (2 P.)

$$\hat{h}(\omega) = \sqrt{2\pi}\hat{g}(\omega) \cdot \hat{g}(\omega) = \sqrt{2\pi}(\hat{f}(\omega))^2$$

$g(x) = \mathrm{e}^{-2x^2}$, i.e. the given function f with $a = 2$. (1 P.)

From the formula sheet we know $\hat{g}(\omega) = \frac{1}{2}\mathrm{e}^{-\frac{\omega^2}{8}}$ (1 P.)

So we obtain for $\hat{h}$ (2 P.)

$$\hat{h}(\omega) = \sqrt{2\pi}\left(\frac{1}{2}\mathrm{e}^{-\frac{\omega}{8}}\right)^2 = \sqrt{2\pi}\frac{1}{4}\mathrm{e}^{-\frac{\omega^2}{4}} = \frac{2}{4}\sqrt{\pi}\frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\omega^2}{4}}$$

with is up to the first two factors the function f with $a = 1$ (1 P.)

$$\hat{h}(\omega) = \frac{\sqrt{\pi}}{2}\mathcal{F}\left(\mathrm{e}^{-x^2}\right)$$

and hence $h(x) = \frac{\sqrt{\pi}}{2}\mathrm{e}^{-x^2}$. (1 P.)

Problem 8. (Numerical Methods for Ordinary Differential Equations, 12 points)

To solve a general first-order ordinary equation of the form

$$y'(t) = f(t, y(t)) \quad \text{for } t > t_0, \quad y(t_0) = y_0$$

numerically, we consider the explicit Runge-Kutta method known as **Ralston's method with 3 stages** which is given by the Butcher tableau

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 \\ \hline & 2/9 & 1/3 & 4/9 \end{array}$$

a) Determine the consistency order of this Runge-Kutta method.

b) Now complete all gaps indicated by ... in the following Python code snippet to provide an implementation of Ralston's method. Assume a uniform time-step size. Arguments passed the `rkm` function argument are

- `y0` : initial value
- `t0` : initial time
- `T` : final time
- `f` : right-hand side of the ordinary differential equation
- `Nmax`: number of time-steps

```
import numpy as np

def rkm(y0, t0, T, f, Nmax):
    ts = [t0]
    ys = [y0]
    dt = ...

    while (ts[-1] < T):
        t, y  = ts[-1], ys[-1]

        k1 = ...
        k2 = ...
        k3 = ...

        ys.append(...)
        ts.append(...)

    return np.array(ts), np.array(ys)
```

c) Next, consider another explicit Runge-Kutta method, this time given by the Butcher tableau

$$\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 \\
3/4 & 0 & 3/4 & 0 & 0 \\
1 & 2/9 & 1/3 & 4/9 & 0 \\
\hline
 & 7/24 & 1/4 & 1/3 & 1/8
\end{array}$$

This Runge-Kutta method is known to have consistency order 4 and can be combined with Ralston's method to devise an *adaptive* Runge-Kutta method. Write down the final Butcher tableau for the resulting **adaptive embedded** Runge-Kutta method and give a short explanation of how you found the final Butcher tableau.

Solution.

a) A general Runge–Kutta method is described by the tableau

$$\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \dots & a_{1s} \\
c_2 & a_{21} & a_{22} & \dots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_s & a_{s1} & a_{s2} & \dots & a_{ss} \\
\hline
 & b_1 & b_2 & \dots & b_s
\end{array}$$

and the order p of consistency can be determined by verifying the conditions below (see formula sheet):

p	Conditions
1	$\sum_{i=1}^{s} b_i = 1$
2	$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$
3	$\sum_{i=1}^{s} b_i {c_i}^2 = \frac{1}{3}$ $\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$
4	$\sum_{i=1}^{s} b_i {c_i}^3 = \frac{1}{4}$ $\sum_{i=1}^{s} \sum_{j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$ $\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} {c_j}^2 = \frac{1}{12}$ $\sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$

The method will be consistent of order p if, and only if, all conditions up to the p-th row are fulfilled. For Ralston's method, we have

$$\sum_{i=1}^{s} b_i = \frac{2}{9} + \frac{1}{3} + \frac{4}{9} = 1$$

$$\sum_{i=1}^{s} b_i c_i = \frac{2}{9} \times 0 + \frac{1}{3} \times \frac{1}{2} + \frac{4}{9} \times \frac{3}{4} = \frac{1}{2}$$

$$\sum_{i=1}^{s} b_i {c_i}^2 = \frac{2}{9} \times 0^2 + \frac{1}{3} \times \left(\frac{1}{2}\right)^2 + \frac{4}{9} \times \left(\frac{3}{4}\right)^2 = \frac{1}{2}$$

$$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = b_1(a_{11}c_1 + a_{12}c_2 + a_{13}c_3) + b_2(a_{21}c_1 + a_{22}c_2 + a_{23}c_3) + b_3(a_{31}c_1 + a_{32}c_2 + a_{33}c_3)$$

$$= \frac{2}{9} \times 0 + \frac{1}{3}\left(\frac{1}{2} \times 0\right) + \frac{4}{9}\left(0 \times 0 + \frac{3}{4} \times \frac{1}{2} + 0 \times \frac{3}{4}\right) = \frac{1}{6}$$

$$\sum_{i=1}^{s} b_i {c_i}^3 = \frac{2}{9} \times 0^3 + \frac{1}{3} \times \left(\frac{1}{2}\right)^3 + \frac{4}{9} \times \left(\frac{3}{4}\right)^3 = \frac{11}{48} \neq \frac{1}{4}.$$

Therefore, the method is third-order consistent.

b)

```python
import numpy as np

def rkm(y0, t0, T, f, Nmax):
    ts = [t0]
    ys = [y0]
    dt = (T-t0)/Nmax

    while (ts[-1] < T):
        t, y  = ts[-1], ys[-1]

        k1 = f(t,y)
        k2 = f(t+1/2*dt,y+1/2*dt*k1)
        k3 = f(t+3/4*dt,y+dt*3/4*k2)

        ys.append(y + dt/9*(2*k1+3*k2+4*k3))
        ts.append(t + dt)

    return np.array(ts), np.array(ys)
```

c) For any given Butcher coefficients c_4 and $\{a_{4j}\}_{j=1}^{4}$, Ralston's method can formally be written as a 4-stage Runge-method by simply setting the 4th weight b_4 to 0, thus ignoring any information from the 4th stage computation. Thus the 3rd order 3 stage Ralston method

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 \\ \hline & 2/9 & 1/3 & 4/9 \end{array}$$

is equivalent to

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 & 0 \\ 1 & 2/9 & 1/3 & 4/9 & 0 \\ \hline & 2/9 & 1/3 & 4/9 & 0 \end{array}$$

Combining this table with the 4th order table from c) we obtain the final Butcher tableau for the embedded Runge-Kutta method:

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 & 0 \\ 1 & 2/9 & 1/3 & 4/9 & 0 \\ \hline & 2/9 & 1/3 & 4/9 & 0 \\ \hline & 7/24 & 1/4 & 1/3 & 1/8 \end{array}$$

Problem 9. (Heat equation, 12 pts)

Consider the following heat equation

$$u_t(x,t) = c^2 u_{xx}(x,t), \quad t > 0, x \in (0, 4\pi)$$

with boundary conditions

$$u(0,t) = u(4\pi, t) = 0, \quad t > 0$$

and initial condition

$$u(x,0) = \max\{0, -\sin(x/2)\}, \quad x \in (0, 4\pi).$$

a) Show that the Fourier sine series solution of this above heat equation with boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{nx}{4}\right) e^{-\frac{c^2 n^2 t}{16}}$$

with real numbers B_n, $n \geq 1$, by using the separation of variables method.

b) Compute the Fourier sine series solution of the above heat equation with the given boundary conditions and initial conditions. Write down the three first non-zero terms of the solution.
Hint: You may use the following identities:

$$\sin(a+b)\sin(a-b) = \sin^2(a) - \sin^2(b) \tag{1}$$

$$\int \sin^2(kx)\,\mathrm{d}x = \frac{x}{2} - \frac{\sin(2kx)}{4k} + \text{constant} \quad \text{for some } k \in \mathbb{R}. \tag{2}$$

Solution.

a) We set

$$u(x,t) = F(x)G(t).$$

This gives

$$F(x)G'(t) = c^2 F''(x)G(t).$$

Separation of variables leads to

$$\frac{G'}{c^2 G} = \frac{F''}{F}.$$

As the left-hand side depends only on t and the right-hand side only on x, both fractions must be equal to a constant, say k. For $k \geq 0$ we get the trivial solution $u \equiv 0$. Therefore, $k < 0$, and we set $k = -p^2$. We get the two ODEs:

$$F'' + p^2 F = 0$$
$$G' + c^2 p^2 G = 0.$$

The first ODE has the general solution

$$F(x) = A\cos(px) + B\sin(px).$$

Using the boundary conditions, we get

$$u(0,t) = F(0)G(t) = 0 = u(4\pi,t) = F(4\pi)G(t).$$

This gives $F(0) = F(4\pi) = 0$, as otherwise we would get $G(t) \equiv 0$. Then $F(0) = A = 0$ and $F(4\pi) = B\sin(4p\pi) = 0$ with $B \neq 0$, thus $p = \frac{n\pi}{4\pi} = \frac{n}{4}$, $n = 1, 2, \ldots$. We can set $B = 1$ and obtain $F_n(x) = \sin\left(\frac{nx}{4}\right)$. The second ODE has the form (with $p = \frac{n}{4}$)

$$G' + c^2 p^2 G = G' + (cn/4)^2 G = 0.$$

Its general solution is

$$G_n = B_n e^{-(cn/4)^2 t}$$

Hence (for $n = 1, 2, 3, \ldots$), the function

$$u_n(x,t) = F_n G_n = B_n \sin\left(\frac{nx}{4}\right) e^{-(cn/4)^2 t}$$

solves the heat equation with the given boundary conditions. Therefore (i.e. because of the superposition principle), the series

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{nx}{4}\right) e^{-\frac{c^2 n^2 t}{16}}$$

is also a solution of the problem.

b) The solution of the problem is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{nx}{4}\right) e^{-\frac{c^2 n^2 t}{16}}.$$

The initial condition gives

$$u(x,0) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{nx}{4}\right) = \max(0, -\sin(x/2))$$
$$= \begin{cases} 0 & \text{if } 0 \le x \le 2\pi, \\ -\sin(x/2) & \text{if } 2\pi < x \le 4\pi \end{cases}$$

with

$$\begin{aligned} B_n &= \frac{2}{4\pi} \int_0^{4\pi} \max\left(0, -\sin(x/2)\right) \sin\left(\frac{nx}{4}\right) dx \\ &= \frac{1}{2\pi} \int_{2\pi}^{4\pi} -\sin(x/2) \cdot \sin(nx/4) dx. \end{aligned}$$

For even n, this is zero (shift interval around zero and cf. lecture)

$$\begin{aligned} &\overset{n \text{ odd}}{=} \frac{-1}{2\pi} \cdot \int_{2\pi}^{4\pi} \sin\left(\frac{(n+2)x}{8} - \frac{(n-2)x}{8}\right) \cdot \sin\left(\frac{(n+2)x}{8} + \frac{(n-2)x}{8}\right) dx \\ &\overset{(1)}{=} \frac{-1}{2\pi} \cdot \int_{2\pi}^{4\pi} \sin^2\left(\frac{n+2}{8}x\right) - \sin^2\left(\frac{n-2}{8}x\right) dx \\ &\overset{(2)}{=} \frac{-1}{2\pi} \cdot \left[\frac{x}{2} - \frac{\sin(2(n+2)x/8)}{4\frac{n+2}{8}} - \frac{x}{2} + \frac{\sin(2(n-2)x/8)}{4\frac{n-2}{8}}\right]_{2\pi}^{4\pi} \\ &= \frac{-1}{\pi} \cdot \left[\frac{\sin((n-2)x/4)}{n-2} - \frac{\sin((n+2)x/4)}{n+2}\right]_{2\pi}^{4\pi} \\ &= \frac{-1}{\pi} \cdot \left(\frac{(-1)^{\frac{n-1}{2}}}{n-2} - \frac{(-1)^{\frac{n-1}{2}}}{n+2}\right), \end{aligned}$$

where we only needed the lower integration bounds in the last step. Therefore, we have

$$B_1 = \frac{-8}{3\pi}, \quad B_2 = 0, \quad B_3 = \frac{8}{5\pi}, \quad B_4 = 0, \quad B_5 = \frac{-8}{21\pi}, \dots$$

Finally, we have

$$\begin{aligned} u(x,t) &= \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx/4) e^{-c^2 n^2 t/16} \\ &= \frac{8}{\pi}\left(-\frac{\sin(x/4)}{3} e^{-c^2 t/16} + \frac{\sin(3x/4)}{5} e^{-9c^2 t/16} - \frac{\sin(5x/4)}{21} e^{-25c^2 t/16} \pm \dots\right) \end{aligned}$$

Formula Sheet. TMA4125/30/35 Matematikk 4N/D, Summer 2022.

Fourier Transform. The Fourier Transform $\hat{f} = \mathcal{F}(f)$ and its inverse $f = \mathcal{F}^{-1}(\hat{f})$ are

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \quad \text{and} \quad f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega$$

Laplace Transform. The Laplace transform $F(s)$ of $f(t)$, $t \geq 0$, reads

$$F(s) = \int_0^{\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

List of Fourier Transforms.

$f(x)$	$\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}\mathrm{e}^{-\frac{\omega^2}{4a}}$
$\mathrm{e}^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{a}{\omega^2+a^2}$
$\frac{1}{x^2+a^2}$ for $a > 0$	$\sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-a\lvert\omega\rvert}}{a}$
$\begin{cases} 1 & \text{for } \lvert x\rvert < a \\ 0 & \text{otherwise.}\end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega a)}{\omega}$

List of Laplace Transforms.

$f(t)$	$F(s)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, see Note[(a)]
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$\mathrm{e}^{-sa}F(s)$
$\delta(t-a)$	e^{-sa}

[(a)] where for $n \in \mathbb{N}$ we have $\Gamma(n+1) = n!$

Trigonometric identities.

- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha-\beta) + \sin(\alpha+\beta)\big)$
- $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$
- $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

We also discussed the sinus cardinalis $\operatorname{sinc}(x) = \frac{\sin x}{x}$.

Fourier Series. For a 2π-periodic function f we can write

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{ikx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

with coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \quad n = 0,1,2,\dots, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x, \quad n = 1,2,\dots,$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-ikx}\,\mathrm{d}x, \quad k \in \mathbb{Z}.$$

Order conditions for Runge-Kutta methods

p	Conditions
1	$\sum_{i=1}^{s} b_i = 1$
2	$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$
3	$\sum_{i=1}^{s} b_i {c_i}^2 = \frac{1}{3}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$
4	$\sum_{i=1}^{s} b_i {c_i}^3 = \frac{1}{4}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} {c_j}^2 = \frac{1}{12}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$