Exercise #1

17. January 2022

Problem 1. (Vector space, inner product space)

a) Show that the set of functions

$$V := \text{span}\{1, \sin(x), \cos(x)\} = \{u(x) : u(x) = a \cdot 1 + b\sin(x) + c\cos(x), \quad a, b, c \in \mathbb{R}\}$$

defined on $[0, 2\pi]$ is a vector space.

b) Show that $\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)dx$ is an inner product on V.

c) Show that the three functions

$$f_1(x) = \frac{1}{\sqrt{2\pi}}, \quad f_2(x) = \frac{1}{\sqrt{\pi}}\sin(x), \quad \text{and } f_3(x) = \frac{1}{\sqrt{\pi}}\cos(x)$$

constitute an orthonormal basis for V.
Hint. The two integrals might be useful $\int_0^{2\pi} \sin^2(x)\mathrm{d}x = \int_0^{2\pi} \cos^2(x)\mathrm{d}x = \pi$

d) (optional) Compute the projection of $g(x) = x$ onto V.
Hint. The two integrals $\int_0^{2\pi} x\sin(x)\mathrm{d}x = -2\pi$ and $\int_0^{2\pi} x\cos(x)\mathrm{d}x = 0$ might be helpful.

Problem 2. (Norms)

Consider the space, $C[0,1]$, of continuous functions defined on $[0,1]$. Verify that,

$$\|f\| := \frac{1}{2}\left(\max_{x\in[0,0.5]} |f(x)| + \max_{x\in[0.5,1]} |f(x)| \right)$$

is a norm on $C[0,1]$.

Problem 3. (Gram-Schmidt)

a) Let $V = \text{span}\{a_1, a_2, a_3\}$, where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathbf{a}_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \qquad \mathbf{a}_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Verify that the dimension of V is 3.
Hint. this is the same as checking that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are linearly independent. This is the case if the matrix $\begin{pmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{pmatrix}$ has full column rank.

b) Perform the Gram-Schimdt procedure for the vectors $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ in order to obtain an *orthonormal* basis for V. Plot this orthonormal basis, as vectors pointing out from the origin.

Hint. You can use the following code to plot vectors in $\mathbb{R}^3$:

```
import numpy as np
import matplotlib.pyplot as plt

a1 = np.array([1., 0., 0.])
a2 = np.array([0., 1., 0.])
a3 = np.array([0., 0., 1.])

fig = plt.figure()
ax = fig.add_subplot(projection='3d')

ax.set_xlim([-1.,1.])
ax.set_ylim([-1.,1.])
ax.set_zlim([-1.,1.])

ax.quiver(0, 0, 0, a1, a2, a3, color='blue')
```

Note that this colors all 3 vectors in blue. You can use multiple such `quivers` to visualize the occurring bases in different colors.

Furthermore, to compute the dot product of two `numpy` arrays, you may use `np.dot(a1,a2)`. In order to compute the Euclidean norm of a `numpy` array, you may use `np.linalg.norm(a1)`.

Problem 4. (Taylor Polynomials)

a) Compute all Taylor polynomials of $f(x) = x^4 + 2x^3 + x^2 + 5$ around $x_0 = 1$

b) Compute the Taylor series of $g(x) = \ln(1 + x)$ around $x_0 = 0$