Formula Sheet. TMA4125 Matematikk 4N, Vår 2022.

Fourier Transform. The Fourier Transform $\hat{f} = \mathcal{F}(f)$ and its inverse $f = \mathcal{F}^{-1}(\hat{f})$ are

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x \quad \text{and} \quad f(x) = \mathcal{F}^{-1}(\hat{f})(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{\mathrm{i}\omega x}\,\mathrm{d}\omega$$

Laplace Transform. The Laplace transform $F(s)$ of $f(t)$, $t \geq 0$, reads

$$F(s) = \int_{0}^{\infty} \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

List of Fourier Transforms.

$f(x)$	$\hat{f}(\omega)$
e^{-ax^2}	$\frac{1}{\sqrt{2a}}\mathrm{e}^{-\frac{\omega^2}{4a}}$
$\mathrm{e}^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{a}{\omega^2+a^2}$
$\frac{1}{x^2+a^2}$ for $a > 0$	$\sqrt{\frac{\pi}{2}}\frac{\mathrm{e}^{-a\lvert\omega\rvert}}{a}$
$\begin{cases} 1 & \text{for } \lvert x\rvert < a \\ 0 & \text{otherwise.} \end{cases}$	$\sqrt{\frac{2}{\pi}}\frac{\sin(\omega a)}{\omega}$

List of Laplace Transforms.

$f(t)$	$F(s)$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, see Note[a]
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$\mathrm{e}^{-sa}F(s)$
$\delta(t-a)$	e^{-sa}

[a] where for $n \in \mathbb{N}$ we have $\Gamma(n+1) = n!$

Trigonometric identities.

- $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
- $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$
- $\sin\alpha\cos\beta = \frac{1}{2}\big(\sin(\alpha-\beta) + \sin(\alpha+\beta)\big)$
- $\cos(2\alpha) = 2\cos^2(\alpha) - 1 = 1 - 2\sin^2(\alpha)$
- $2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$
- $2\cos\alpha\sin\beta = \sin(\alpha+\beta) - \sin(\alpha-\beta)$
- $2\cos\alpha\cos\beta = \cos(\alpha-\beta) + \cos(\alpha+\beta)$
- $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$

We also discussed the sinus cardinalis $\operatorname{sinc}(x) = \frac{\sin x}{x}$.

Fourier Series. For a 2π-periodic function f we can write

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

with coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x, \quad n = 0,1,2,\ldots, \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x, \quad n = 1,2,\ldots,$$

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x, \quad k \in \mathbb{Z}.$$