NTNU

Norwegian University of Science and Technology

TMA4125 Matematikk 4N

Polynomial interpolation: Error Analysis

Ronny Bergmann

Department of Mathematical Sciences, NTNU.

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Motivation

Given some function $f \in C[a, b]$. Choose $n + 1$ distinct nodes in $[a, b]$ and let $p_n(x) \in \mathbb{P}_n$ satisfy the interpolation condition

$$p_n(x_i) = f(x_i), \qquad i = 0, \dots, n.$$

What can be said about the error $e(x) = f(x) - p_n(x)$?

Aspects.

1. If the polynomial p_n is used to approximate a function f, can we find an expression for the error, or at least an upper bound?
2. How can the error be made as small as possible?

A numerical example

Let's first get a feeling of what to expect

Example. Let

- $n \in \mathbb{N}$ (for example $n = 4, 8, 16$, or 32)
- $f(x) = \sin(x)$, $x \in [0, 2\pi]$
- equidistant nodes $x_i = ih$, $i = 0, \ldots, n$, and $h = 2\pi/n$

Task.

1. Calculate the interpolation polynomial p_n using `cardinal` and `lagrange`
2. plot the error $e_n(x) = f(x) - p_n(x)$. Notice how the error is distributed over the interval.
3. find the maximum error $\max_{x \in [a,b]} |e_n(x)|$ for each n.

A second example

As a second example let's look at Runge's function

$$f(x) = \frac{1}{1+x^2}, \qquad x \in [-5, 5].$$

What do we have to adapt in our code?

We define a new function `f` (given in the notebook even) and have to adapt the interval to $a = -5$ and $b = 5$. All the rest can just be copied.

Conclusion. In Runge's example it seems that with more points the result does not get better as we would expect, but worse.

An idea towards the interpolation error

To find an expression for $e(x) = f(x) - p_n(x)$, let the nodes $x_i \in [a, b]$, $i = 0, \dots, n$ be distinct, and that $f \in C^{n+1}[a, b]$.

From interpolation conditions: error in the nodes, $e(x_i) = 0$.

Consider the function

$$\omega(x) = \prod_{i=0}^{n}(x - x_i) = x^{n+1} + \dots \in \mathbb{P}_{n+1}$$

Choose an arbitrary $x \in [a, b]$, where $x \neq x_i$, $i = 0, 1, \dots, n$.
For this fixed x, define a function in t as:

$$\varphi(t) = e(t)\omega(x) - e(x)\omega(t).$$

An idea towards the interpolation error II

Looking at $\varphi(t) = e(t)\omega(x) - e(x)\omega(t)$ (remember: fixed x) we see

- $\varphi(t) \in C^{(n+1)}$ as well
- $\varphi(t)$ has $n+2$ distinct zeros (all $t = x_i$ and $t = x$).

$\Rightarrow$ by Rolle's theorem: $\varphi'(t)$ has at least $n+1$ distinct zeros (one between each of the zeros of $\varphi(t)$).

$\Rightarrow$ by Rolle's theorem: $\varphi''(t)$ has at least n distinct zeros

- we repeat this until we obtain that
- $\varphi^{(n+1)}(t)$ has at least one zero in $[a, b]$
  let us call this $\xi(x)$, as it does depend on the fixed x.

This last derivative looks like and at $t = \xi(x)$ we get

$$\begin{aligned}\varphi^{(n+1)}(\xi(x)) &= e^{(n+1)}(\xi(x))\omega(x) - e(x)\omega^{(n+1)}(\xi(x)) \\ &= \left(f^{(n+1)}(\xi(x)) - p_n^{(n+1)}(\xi(x))\right)\omega(x) - \left(f(x) - p_n(x)\right)\omega^{(n+1)}(t)(n+1)!\end{aligned}$$

Interpolation error

Theorem. Given $f \in C^{(n+1)}[a,b]$. Let $p_n \in \mathbb{P}_n$ interpolate f in $n+1$ distinct nodes $x_i \in [a,b]$, $i = 0,\dots,n$. For each $x \in [a,b]$ there is at least one $\xi(x) \in (a,b)$ such that

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x - x_i).$$

Proof. Take $\varphi(t) = e(t)\omega(x) - e(x)\omega(t)$ we constructed earlier and use

$$\varphi^{(n+1)}(\xi(x)) = f^{(n+1)}(\xi(x))\omega(x) - \big(f(x) - p_n(x)\big)(n+1)! = 0$$

and rearrange.

Interpolation error – degrees of freedom

The interpolation error consists of three elements:

- the derivative of the function f
- the number of interpolation points $n + 1$
- and the distribution of the nodes x_i

We cannot do much with the first of these, but we can choose the two others. Let us first look at the most obvious choice of nodes.

Equidistant points

The nodes are equidistant over the interval $[a, b]$ if $x_i = a + ih$, $h = (b-a)/n$. In this case it can be proved that:

$$|\omega(x)| \leq \frac{h^{n+1}}{4} n!$$

such that

$$|e(x)| \leq \frac{h^{n+1}}{4(n+1)} M, \qquad M = \max_{x\in[a,b]} |f^{(n+1)}(x)|.$$

for all $x \in [a, b]$.

Example. Let again $f(x) = \sin(x)$ and $p_n(x)$ the polynomial interpolating $f(x)$ in $n+1$ equidistant points $x_0 = a, x_1, \ldots, x_{n-1}, x_n = b$.
Then $\max_{x\in[0,2\pi]} |f^{(n+1)}(x)| = M = 1$ for all n, so we have for any $x \in [a, b]$

$$|e_n(x)| = |f(x) - p_n(x)| \leq \frac{1}{4(n+1)} \left(\frac{2\pi}{n}\right)^{n+1}.$$

$\Rightarrow$ Compare to code: How close is this?

Idea: Choose other nodes

So how can the error be reduced? Can we choose better points?

Approach. For a given n distribute the nodes $x_0, \ldots, x_n$ in a way to make

$$|\omega(x)| = \prod_{i=0}^{n} |x - x_i|$$

as small as possible (for all $x \in [a, b]$).
We will first do this on a standard interval $[-1, 1]$, and then transfer the results to some arbitrary interval $[a, b]$.

Let us start taking a look at $\omega(x)$ for equidistant nodes on the interval $[-1, 1]$, for different values of n.

Observation. $\max_{x\in[-1,1]} |\omega(x)|$ does become smaller,
but we have peaks near the boundary -1 and 1.

The optimal choice: Chebyshev nodes

Reasoning.
Place more nodes near the boundaries -1 and 1 to avoid the peaks.

It can be proved that the optimal choice are the **Chebyshev nodes** given by

$$\tilde{x}_i = \cos\left(\frac{(2i+1)\pi}{2(n+1)}\right), \qquad i = 0, \dots, n,$$

Theorem. Let $\omega_{\mathrm{Cheb}}(x) = \prod_{i=0}^{n}(x - \tilde{x}_i)$. Then it holds that

$$\frac{1}{2^n} = \max_{x\in[-1,1]} |\omega_{\mathrm{Cheb}}(x)| \leq \max_{x\in[-1,1]} |q(x)|$$

for all polynomials $q \in \mathbb{P}_n$ such that
$q(x) = x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$.

Generalization to intervals $[a, b]$

The distribution of nodes can be transferred to an interval $[a, b]$ by the linear transformation

$$x = \frac{b-a}{2}\tilde{x} + \frac{b+a}{2}$$

where $x \in [a, b]$ and $\tilde{x} \in [-1, 1]$.

By doing so we get

$$\omega(x) = \prod_{i=0}^{n}(x - x_i) = \left(\frac{b-a}{2}\right)^{n+1} \prod_{i=0}^{n}(\tilde{x} - \tilde{x}_i) = \left(\frac{b-a}{2}\right)^{n+1} \omega_{Cheb}(\tilde{x}).$$

This will give us the lowest possible maximum error on $[a, b]$.

(Improved) Interpolation error estimate

From the theorem on interpolation errors we can conclude:

Theorem. (interpolation error for Chebyshev interpolation) Given $f \in C^{(n+1)}[a, b]$, and let $M_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)|$. Let $p_n \in \mathbb{P}_n$ interpolate f in $n+1$ Chebyshev-nodes $x_i \in [a, b]$. Then

$$\max_{x \in [a,b]} |f(x) - p_n(x)| \leq \frac{(b-a)^{n+1}}{2^{2n+1}(n+1)!} M_{n+1}.$$

In the notebook you can find the function computing the Chebyshev nodes as `chebyshev_nodes(a, b, n)`.

Exercise. Newton polynomials & Runge‘s function

Exercise.

1. Plot $\omega_{\mathrm{Cheb}}(x)$ for $3, 5, 9, 17$ interpolation points.
2. Repeat the interpolation with Runge‘s function

$$f(x) = \frac{1}{1 + x^2}, \quad x \in [-5, 5]$$

   using Chebyshev nodes and compare the result to the equidistant nodes case.

Compare to: Error analysis for Taylor polynomials

Remember that for the nth Taylor polynomial of f around $x_0 \in (a, b)$ we have

$$T_{x_0}^n f(x) := \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k \in \mathbb{P}_n$$

It interpolates the $f'(x_0), f''(x_0), \ldots, f^{(n)}(x_0)$!
Even more: For the error we know

$$f(x) - T_{x_0} f(x) = R_{n+1}(x_0), \quad \text{where } R_{n+1}(x_0) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

where we only now that ξ is between x and x_0.
But we can again bound the derivative as

$$|f(x) - T_{x_0}^n f(x)| \leq \frac{M_{n+1}}{(n+1)!} h^{n+1}$$

where

$$M_{n+1} = \max_{x \in [a,b]} |f^{(n+1)}(x)| \quad \text{and} \quad h = |x - x_0|$$

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