NTNU

Norwegian University of Science and Technology

TMA4125 Matematikk 4N

The Laplace Transform I

Ronny Bergmann

Department of Mathematical Sciences, NTNU.

February 8, 2022

Introduction.

Goal. Solve ordinary differential equations (ODEs).

Typical examples are first order ODEs of the form:
Find a function $y = y(t)$ that fulfils

$$\begin{cases} y'(t) + ay(t) = r(t), & t > 0 \\ y(0) = K_0. \end{cases}$$

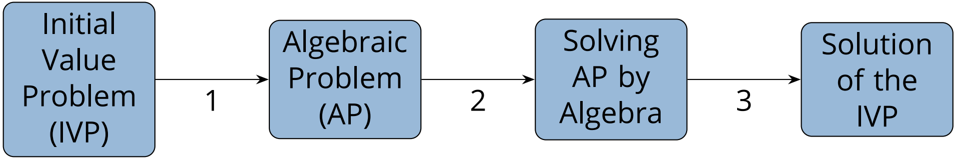
appearing for example in model grow or decay processes.
A second very important example are second order ODEs of the form

$$\begin{cases} y''(t) + ay'(t) + by(t) = r(t), & t > 0 \\ y(0) = K_0, \\ y'(0) = K_1. \end{cases}$$

This form is also called Initial Value Problem (IVP)

General scheme to solve an ODE

Solving an ODE using the Laplace transform consists of the following steps



1. The given ODE is transformed into an algebraic equation, called subsidiary equation
2. The subsidiary equation is solved nby purely algebraic manipulations
3. The solution is transformed back, resulting in the solution of the given problem.

Transform – the general idea

A transform turns a given function f into another function.

Known transforms.

The derivative operator D takes a differentiable function $f\colon [a,b] \to \mathbb{R}$ and assigns/returns a new function $(Df)(x) = f'(x)$.

The Integral I takes a continuous function $f\colon [a,b] \to \mathbb{R}$ and assigns/returns a new function

$$I[f](t) := F(t) = \int_0^t f(x)\,\mathrm{d}x.$$

The multiplication operator M_φ multiplies any given function $f\colon [a,b] \to \mathbb{R}$ by a fixed function $\varphi\colon [a,b] \to \mathbb{R}$

$$M_\varphi f(t) = \varphi(t) \cdot f(t).$$

Roadmap

- Definition of the Laplace transform
- Examples and properties (esp. s-shifting)
- Existence and uniqueness
- derivatives and t-shifting
- Dirac and Heaviside function
- Convolution and Integral Equations
- Solving ODEs using the Laplace transform

The Laplace transform

Definition. Let $f(t)$ be a function that is defined for all $t \geq 0$.
Then the Laplace transform[1] $\mathcal{L}(f)$ of f is a function of a new variable s and defined by

$$F(s) = \mathcal{L}(f) = \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

assuming that the integral exists.

Remark. Sine the Laplace transform is defined using an improper integral, we have to compute it by tracking the limit

$$\int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_0^T \mathrm{e}^{-st} f(t)\,\mathrm{d}t.$$

We denote by $\mathcal{L}^{-1}(F) = f$ the inverse Laplace transform that maps F to f.

[1] Pierre Simon Marquis de Laplace (1749-1827)

Three examples.

Example 1. The Laplace transform of $f_1(t) = c$, $c \in \mathbb{R}$, for $t \geq 0$ is

$$\mathcal{L}(c) = \int_0^\infty \mathrm{e}^{-st} f_1(t)\,\mathrm{d}t = \lim_{T\to\infty} \int_0^\infty \mathrm{e}^{-st} c\,\mathrm{d}t = \lim_{T\to\infty} \frac{c}{-s}\mathrm{e}^{-st}\Big|_0^T = \frac{c}{s} \quad \text{for } s > 0$$

Example 2. The Laplace transform of $f_2(t) = \mathrm{e}^{at}$, $a \in \mathbb{R}$, for $t \geq 0$ is

$$\mathcal{L}(\mathrm{e}^{at}) = \int_0^\infty \mathrm{e}^{-(s-a)t}\,\mathrm{d}t = \lim_{T\to\infty} \frac{1}{a-s}\mathrm{e}^{-(s-a)t}\Big|_0^T = \frac{1}{s-a} \quad \text{for } s - a > 0$$

Example 3. Trying to compute the Laplace transformation of $f_3(t) = \mathrm{e}^{t^2}$:

$$\mathcal{L}(\mathrm{e}^{t^2}) = \int_0^\infty \mathrm{e}^{t^2-st}\,\mathrm{d}t$$

which does not exist, since e^{t^2} increases much faster than e^{-st} decreases. Thus $\lim_{t\to\infty} e^{t^2-st} = \infty$ and the integral does not exist

Linearity

Let f, g be two functions whose Laplace transforms exist. Let $a, b \in \mathbb{R}$. Then we have

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t)),$$

i. e. the Laplace transform is linear.

Proof. Since both Laplace transforms of f and g exists, this follows directly from the linearity of integration.

Laplace transform of sine & cosine hyperbolicus

Example 4. Find the transforms of $\cosh(at)$ and $\sinh(at)$

The cosine hyperbolicus reads $\cosh(at) = \frac{1}{2}\left(e^{at} + e^{-at}\right)$ and the sine hyperbolicus $\sinh(at) = \frac{1}{2}\left(e^{at} - e^{-at}\right)$

Laplace transform of sine & cosine

Find the transforms of $\cos(t)$ and $\sin(t)$

The cosine reads $\cos(\omega t) = \frac{1}{2}\left(\mathrm{e}^{\text{ß}\omega t} + \mathrm{e}^{-\text{ß}\omega t}\right)$ and the sine $\sin(\omega t) = \frac{1}{2}\left(\mathrm{e}^{\text{ß}\omega t} - \mathrm{e}^{-\text{ß}\omega t}\right)$

Piecewise continuous functions

NTNU

Definition. A function $f\colon [0,\infty) \to \mathbb{C}$ is called pieceweise continuous if f fulfils the following properties

- on every finite interval $[a,b]$, $0 \leq a < b < \infty$ exists a partition $a = x_0 < t_1 < \cdots, t_n = b$ such that $f\big|_{(t_i,t_{i+1})}$ is continuous, $i = 0,\dots,n-1$
- $\lim\limits_{t\to t_i^+} f(t)$ exists
- $\lim\limits_{t\to t_i^-} f(t)$ exists

Existence of the Laplace transform

Idea. We have to make sure the growth is not too large

Theorem. Let $f\ [0,\infty) \to \mathbb{C}$

- be piecewise continuous
- and f be "upper bounded in growth", i. e.
  there exists $M > 0$ and $a > 0$ such that

$$|f(t)| \le M\mathrm{e}^{at} \quad \text{for } t \ge 0.$$

Then the Laplace transform $\mathcal{L}(f) = F(s)$ is well-defined for $s > a$.

Uniqueness of the Laplace transform

Theorem. Let f and g be piecewise continuous. If

$$\mathcal{L}(f) = \mathcal{L}(g)$$

holds, then we have

$$f = g$$

everywhere where both f and g are continuous.

Some Laplace transforms

	$f(t)$	$\mathcal{L}(f)$
1	1	$\frac{1}{s}$
2	t	$\frac{1}{s^2}$
3	t^2	$\frac{2!}{s^3}$
4	$t^n,\ n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$
5	$t^\alpha,\ \alpha > 0$	$\frac{\Gamma(a+1)}{s^{a+1}}$
6	e^{at}	$\frac{1}{s-a}$

	$f(t)$	$\mathcal{L}(f)$
1	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
2	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
3	$\cosh(at)$	$\frac{s}{s^2-a^2}$
4	$\sinh(at)$	$\frac{a}{s^2-a^2}$
1	$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
2	$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$

A short example

NTNU

Example 6. Compute the Laplace transform of $f(t) = 5t^3 - 2\mathrm{e}^t$

First shifting theorem, s-shifting

Theorem. Let $f(t)$ be given with Laplace transform $F(s)$ (for all $s > k$ for some k)
Then the function $\mathrm{e}^{at}f(t)$ has the Laplace transform $F(s-a)$ for $s - a > k$.

In short the s shift is given by

$$\mathcal{L}(\mathrm{e}^{at}f(t)) = F(s-a)$$

Proof.

Example for shifting to find the inverse transform

Example 7. Find the inverse of the transform of (i. e. reconstruct f from)

$$\mathcal{L}(f) = \frac{3s - 137}{s^2 + 2s + 401}$$

Laplace transform and derivatives

Theorem. Let $f : [0, \infty) \to \mathbb{R}$ (or $\mathbb{C}$) such that

- it is differentiable
- fulfils the growth condition
- and its derivative f' is piecewise continuous.

Then we have

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0).$$

Proof.

Laplace transform and derivatives II

Idea. If f is “nice enough”, we can generalize this easily.

Corollary. If all derivatives $f, f', f'', \dots, f^{(n-1)}$ fulfil the growth condition and $f^{(n)}$ is piecewise continuous, we obtain

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

Note. The theorem provides that we can use the Laplace transform to … transform

$$f^{(n)} \xrightarrow{\mathcal{L}} s^nF(s) + p(s),$$

where $p(s) \in \mathbb{P}^{n-1}$ is a polynomial of order $n-1$ in s.
The values of $f, f', f'', \dots, f^{(n-1)}$ at 0 is exactly what our IVP provides!

Solving an IVP using the Laplace transform

Given our IVP from the very first slide

$$\begin{cases} y''(t) + ay'(t) + by(t) = r(t), & t > 0 \\ y(0) = K_0, \\ y'(0) = K_1. \end{cases}$$

Question. How can we now solve this (using Laplace)?

Solving an IVP with Laplace – Recipe

To solve an IVP using the Laplace transform, we have to

1. compute $R(s) = \mathcal{L}(r)$ ("Input")
2. set up the transfer function $Q(s)$
3. (simple case) homogeneous initial conditions $K_0 = K_1 = 0$

$\Rightarrow$ $\mathcal{L}(y) = Q(s)R(s)$ and Q only involves a and b

3. (general case) we have to reorganise the subsidiary equation

$$Y(s) = Q(s)R(s) + Q(s)((s+1)K_0 + K_1)$$

4. compute $y(t) = \mathcal{L}^{-1}(Y(s))$ ("Output")

Note. Three main steps: Laplace transform of r, set up subsidiary equation and rearrange, inverse Laplace transform

Multiplication Theorem

An analogue of the derivation theorem is the multiplication theorem.

Theorem. Let f be piecewise continuous

$$\mathcal{L}(tf(t)) = -\frac{\mathrm{d}}{\mathrm{d}s}\mathcal{L}(f) = -F'(s)$$

Proof.

Example of the multiplcation theorem

Example. Compute the Laplace transform of $t\sin(t)$

Exercise. Try yourself to compute $\mathcal{L}(t^n\sin(t)) = \mathcal{L}(t(t^{n-1}\sin(t)))$

Laplace transform and integration

Theorem. Let f be piecewise continuous and fulfil our growth condition.

We define $g(t) := \int_0^t f(\tau)\,\mathrm{d}\tau$

Then it holds

$$\mathcal{L}(g(t)) = \mathcal{L}\Big(\int_0^t f(\tau)\,\mathrm{d}\tau\Big) = \frac{1}{s}F(s).$$

Proof.

A known example.

We illustrate the Theorem from last slide with

$$\sin(\omega t) = a \int_0^t \cos(\omega \tau)\, \mathrm{d}\tau \quad \left(\text{since } \frac{\mathrm{d}}{\mathrm{d}t} \sin(\omega t) = \omega \cos(\omega t) \right)$$

To confirm $\mathcal{L}(\sin(\omega t)) = \dfrac{\omega}{s^2 + \omega^2}$ starting from $\mathcal{L}(\cos(\omega t)) = \dfrac{s}{s^2 + \omega^2}$

A complete Example

Solve the initial value problem

$$\begin{cases} y''(t) + y'(t) + 9y(t) = 0, & t > 0 \\ y(0) = 0.16, \\ y'(0) = 0. \end{cases}$$