TMA4125 Matematikk 4N

The Laplace Transform II

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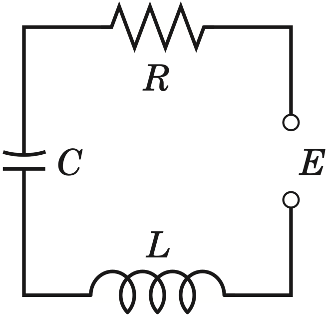
An Application: Electric Current.



Source: Kreyszig, p. 3, cf. also Sec. 2.9 (p. 93)

We have an RLC circuit with

- resistor R (ohm)
- inductor L (henry)
- capacitor C (farad)
- electromotive force E (Voltage V)

With voltage drops (Spenningsavfall)

$$RI, \qquad LI' = L\frac{\mathrm{d}}{\mathrm{d}t}I, \qquad \frac{Q}{C} = \frac{1}{C}\int I \,\mathrm{d}t.$$

Goal. Current $I(t) = \frac{\mathrm{d}}{\mathrm{d}t}Q$ (ampere) where Q is the charge (coulomb).

Kirchhoff's Current Law: Integro-Differential equation for $I(t)$

$$LI' + RI + \frac{1}{C}\int I \,\mathrm{d}t = V$$

The ODE for RLC circuit

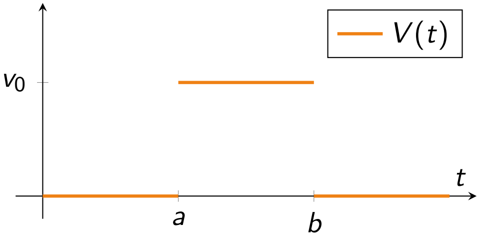
Taking the derivative of

$$LI' + RI + \frac{1}{C}\int I \, \mathrm{d}t = V$$

yields a second order ODE

$$LI'' + RI' + \frac{1}{C}I = V'$$

Question. (or goal for today)

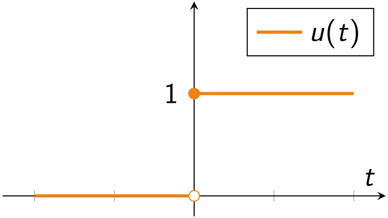


- What happens if the turn on a constant voltage for some time $[a, b]$?
- What is here V'?

The Heaviside function

Definition. (Heaviside function)

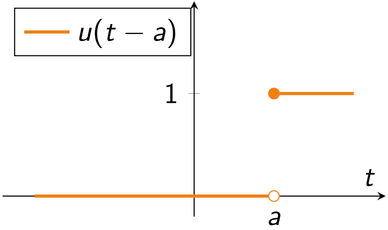
$$u(t) := X_{[0,\infty)}(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0, \end{cases}$$



- For $a \geq 0$:

$$u(t-a) = \tau_a u(t)$$

just shift the Heaviside function to a



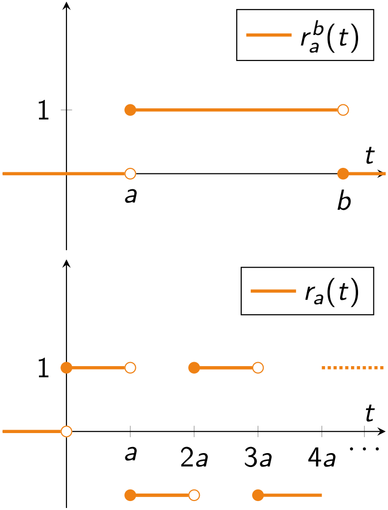
Rectangular functions

- rectangle function:

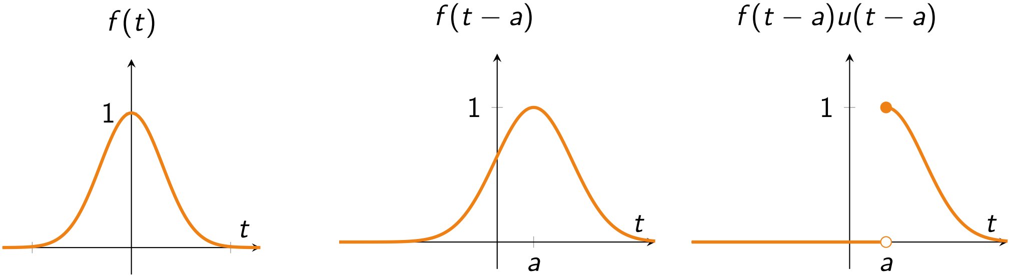
$$r_a^b(t) := u(t-a) - u(t-b), \quad 0 \le a < b$$

- periodic rectangular wave of period $2a$

$$\begin{aligned} r_a(t) &:= u(t) - 2u(t-a) + 2u(t-2a) \\ &\quad - 2u(t-3a) + \dots \\ &= u(t) + \sum_{k=1}^{\infty} (-1)^k 2 r_{ka}^{(k+1)a}(t) \end{aligned}$$



Translation and Cut-off



- if we have a function f (maybe also only defined for $t \geq 0$)
- and we want to shift it, we get $f(t-a)$
- we can “turn on” only at $t = a$

Example. The Laplace transform of the Heaviside function.

$$\mathcal{L}\big(u(t-a)\big) = \int_0^\infty u(t-a)\mathrm{e}^{-st}\,\mathrm{d}t = \int_a^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \frac{1}{-s}\mathrm{e}^{-st}\Big|_{t=a}^{t=\infty} = \frac{\mathrm{e}^{-sa}}{s} = \mathrm{e}^{-sa}\mathcal{L}(1)$$

The t-shift Theorem (the second shift theorem)

Remember. The s-shift Theorem: $\mathcal{L}(\mathrm{e}^{at}f(t)) = F(s-a)$

Theorem. If f has the Laplace transform $F(s)$, then the function

$$g(t) = f(t-a)u(t-a) = \begin{cases} 0 & \text{for } t < a \\ f(t-a) & \text{for } t \geq a \end{cases}$$

has the Laplace transform

$$\mathcal{L}(g(t)) = \mathcal{L}\big(f(t-a)u(t-a)\big) = \mathrm{e}^{-as}F(s)$$

We can also write this (applying $\mathcal{L}^{-1}$ on both sides)

$$f(t-a)u(t-a) = \mathcal{L}^{-1}\big(\mathrm{e}^{-as}F(s)\big).$$

Proof.

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Example.

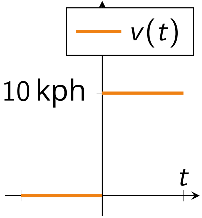
Compute the Laplace transform $\mathcal{L}(f)$ of

$$f(t) = \begin{cases} 0 & \text{for } t < 1 \\ \sin(t-1) & \text{for } t \geq 1. \end{cases}$$

Delta "function" / Dirac "function"

Goal. Find a derivative for the Heaviside function.

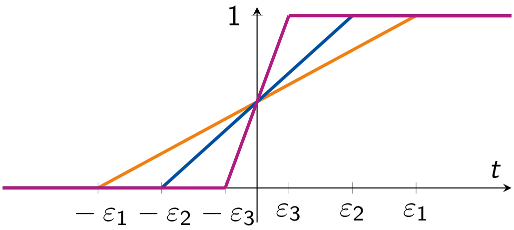
Idea/Motivation. Assume we accelerate a car.
What is immediate acceleration?
What is then $a(t) = \frac{\mathrm{d}}{\mathrm{d}t} v(t) = v'(t)$



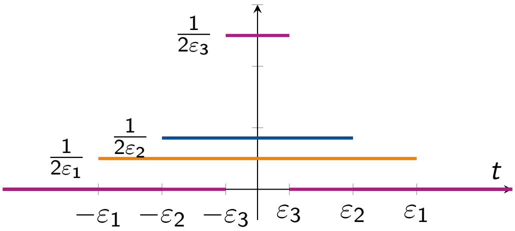
Observation. For $t < 0$ and $t > 0$ we have $v'(t) = 0$. What about $t = 0$?

Trick. Approximate!

speed v_{ε_1}, v_{ε_2}, v_{ε_3},...



acceleration a_{ε_1}, a_{ε_2}, a_{ε_3},...



Delta / Dirac "function" II

Let's fix any ε, then we get

$$\int_{-\infty}^{\infty} a_{\varepsilon}(t)\,\mathrm{d}t = 1$$

independent of ε!

We observe further

- the support of $a_{\varepsilon}(t)$ is an interval of length 2ϵ and gets smaller and smaller for $\epsilon \to 0$
- We know the minimal (0) and maximnal function value.

$$\max_{t\in\mathbb{R}} |a_{\varepsilon}'(t)| = \frac{1}{2\varepsilon}$$

Idea. Taking the limit $\epsilon \to 0$ we obtain a "generalized" function $\delta(t)$

Delta / Dirac "function" III

Taking the limit $\epsilon \to 0$ we obtain a "generalized" function $\delta(t)$ which fulfills

1. $\delta(t) = \begin{cases} \infty & \text{for } x = 0 \\ 0 & \text{for } x \neq 0 \end{cases}$
2. $\int_{-\infty}^{\infty} \delta(t)\,\mathrm{d}t = 1$
3. intuitively: The derivative of the Heaviside function is $u'(t) = \delta(t)$.

This is called (often) Dirac or delta "function".

Obs. From Math 2 we know that $\delta(t)$ can not be a normal function, since its Riemann integral is zero.
So do we know always have to keep the whole construction via $a_\varepsilon(t)$ in mind? No.

Properties of the Delta “function”

If we look at

$$\int_{-\infty}^{\infty} f(t)a_{\varepsilon}(t)\,\mathrm{d}t = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(t)\,\mathrm{d}t \xrightarrow{\varepsilon\to 0} f(0)$$

So we can define the dirac “function” as

$$\int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t = f(0)$$

or more general we arrive at the convolution for some $a \in \mathbb{R}$
(note that δ is an even function)

$$\int_{-\infty}^{\infty} f(t)\delta(t-a)\,\mathrm{d}t = f(a) = \int_{-\infty}^{\infty} f(t)\delta(a-t)\,\mathrm{d}t =: (f * \delta)(a).$$

Dirac distributon

For a reasonable function f, we saw that

$$\int_{-\infty}^{\infty} f(t)a_\varepsilon(t)\,\mathrm{d}t \to f(0) \text{ for } \varepsilon \to 0.$$

And we used this limit in ε to define Dirac “function” $\delta(t)$..

Better: δ can rather be understood as a functional, i. e. a mapping that “returns” a scalar value for every function.

$$f \mapsto \delta(f) := f(0) = \int_{-\infty}^{\infty} f(t)\delta(t)\,\mathrm{d}t$$

Analogon. Measurement of temperature

Measuring temperature...

- can the thought of as a function assigning each point p a temperature in degree.
- can be thought of as something acting on a thermometer

$\Rightarrow$ or: when we probe/measure, we actually get a local average around p

$$\tilde{T} = \int_\Omega T(x)\phi_p(x)\,\mathrm{d}x \quad \text{with} \int \phi_p(x)\,\mathrm{d}x = 1$$

and the support of ϕ_x lies around $p \Rightarrow \phi_p$ is called a test function.

Characterise functions via testing

We denote by

$$C_c^\infty([a,b]) = \Big\{ \phi\colon [a,b] \to \mathbb{R} \,\Big|\, \phi^{(n)}(x) \text{ exists for all } x \in [a,b] \text{ and } \phi^{(n)}(a) = \phi^{(n)}(b) = 0 \text{ for all } n \in \mathbb{N} \Big\}.$$

The set of smooth functions of compact support.

Theorem. Let f, g be two continuous functions defined on $[a,b]$. If

$$\int_a^b f(x)\phi(x)\,\mathrm{d}x = \int_a^b g(x)\phi(x)\,\mathrm{d}x \quad \text{for all } \phi \in C_c^\infty([a,b])$$

then $f \equiv g$.

Characterise derivatives via test functions

For nice functions $f \in C^1$ we can use integration by parts with a test function $\phi \in C_c([a, b])$ and get

$$\int_a^b f'(x)\phi(x)\,\mathrm{d}x = f(x)\phi(x)\Big|_a^b - \int_a^b f(x)\phi'(x)\,\mathrm{d}x = -\int_a^b f(x)\phi'(x)\,\mathrm{d}x$$

(remember that $\phi(a) = \phi(b) = 0$)

$\Rightarrow$ In that sense we can say the dirac distribution is a "generlized derivative" of the Heaviside function u.

The Laplace function of the Dirac "function"

Remember We defined δ by $\int_0^\infty f(t)\delta(t)\,\mathrm{d}t =: \delta(f) =: f(0)$

and for the shiftet delta "function"

$$\int_0^\infty f(t)\delta_a(t) = \int_0^\infty f(t)\delta(t-a)\,\mathrm{d}t = \delta_a(f) = f(a)$$

With these the Laplace transform is easy to see taking $f(t) = \mathrm{e}^{-st}$:

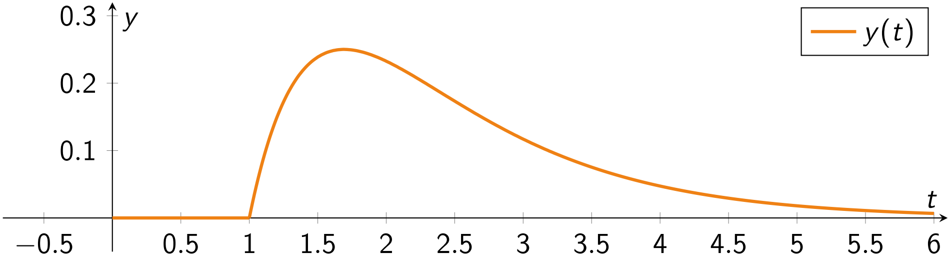
$$\mathcal{L}(\delta_a(t)) = \int_0^\infty \delta(t-a)\mathrm{e}^{-st}\,\mathrm{d}t = \mathrm{e}^{-sa}$$

Hammerblow response of damped Mass-spring system

Example. (cf. Kreyszig, p. 227) Solve

$$\begin{cases} y'' + 3y' + 2y = \delta(t-1) \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

We plot $y(t) = \begin{cases} 0 & \text{for } t < 1 \\ e^{-(t-1)} - e^{-2(t-1)} & \text{for } t \geq 1. \end{cases}$

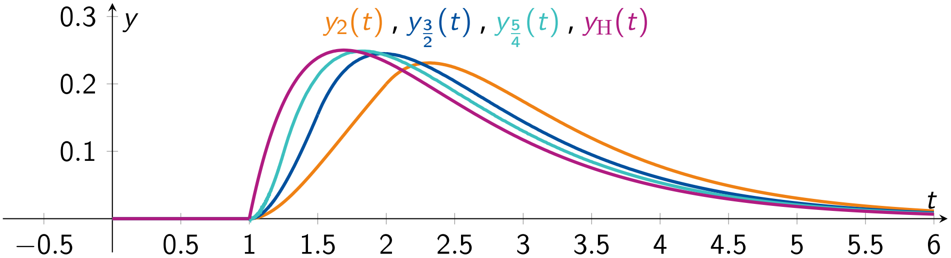


Square Wave response of damped Mass-spring system

Example. (cf. Kreyszig, p. 227) Solve for $a > 1$

$$\begin{cases} y_a'' + 3y_a' + 2y_a = \dfrac{1}{1-a}\big(u(t-1) - u(t-a)\big) \\ y_a(0) = 0, \\ y_a'(0) = 0. \end{cases}$$

$$\Rightarrow y_2(t) = \begin{cases} 0 & \text{for } t < 1, \\ \frac{1}{2} - \mathrm{e}^{-(t-1)} + \frac{1}{2}\mathrm{e}^{-2(t-1)} & \text{for } 1 \le t < 2, \\ -\mathrm{e}^{-(t-1)} + \mathrm{e}^{-2(t-1)} + \frac{1}{2}\mathrm{e}^{-2(t-1)} - \frac{1}{2}\mathrm{e}^{-2(t-2)} & \text{for } t \ge 2. \end{cases}$$



Motivation for convolution

Remember. The Laplace transform is linear: $\mathcal{L}(f+g) = \mathcal{L}(f) + \mathcal{L}(g)$

In the solution of an IVP

$$\begin{cases} y'' + ay' + by = r \\ y(0) = K_0, \\ y'(0) = K_1. \end{cases}$$

We obtained that for $K_0 = K_1 = 0$ we got using $Q(s) = \frac{1}{s^2+as+b}$ that

$$y(t) = \mathcal{L}^{-1}(Y(s)) = \mathcal{L}^{-1}\Big(Q(s)R(s) + Q(s)\big((s+a)K_0 + K_1\big)\Big) = \mathcal{L}^{-1}(Q(s)R(s))$$

Question. What is the inverse Laplace transformation of a product of two functions?

Convolution

Definition. (Laplace version) We define the convolution of f and g as

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,\mathrm{d}\tau$$

if the integral is defined.

Example. for $f(t) = g(t) = \sin(t)$ we get

$$(\sin * \sin)(t) = \frac{1}{2}(\sin t - t\cos t)$$

Example. for $f(t) = g(t) = \sin(t)$ we get

$$(t * 1)(t) = \int_0^t \tau 1\,\mathrm{d}\tau = \frac{1}{2}t^2$$

Convolution Theorem

Theorem. Let f and g be two functions such that $\mathcal{L}(f)$ and $\mathcal{L}(g)$ as well as $\mathcal{L}(f * g)$ exist.

Then it holds

$$\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$$

or equivalently

$$f * g = \mathcal{L}^{-1}\big(F(s)G(s)\big)$$

Proof.

Properties of Convolution

For two functions f, g, h the following properties hold

1. $f * g = g * f$ (commutative)
2. $f * (g + h) = f * g + f * h$ (distributive)
3. $(f * g) * h = f * (g * h)$ (associative)
4. $f * 0 = 0$

Proof. They follow either via integral manipulation or via $\mathcal{L}$

One further example of a convolution

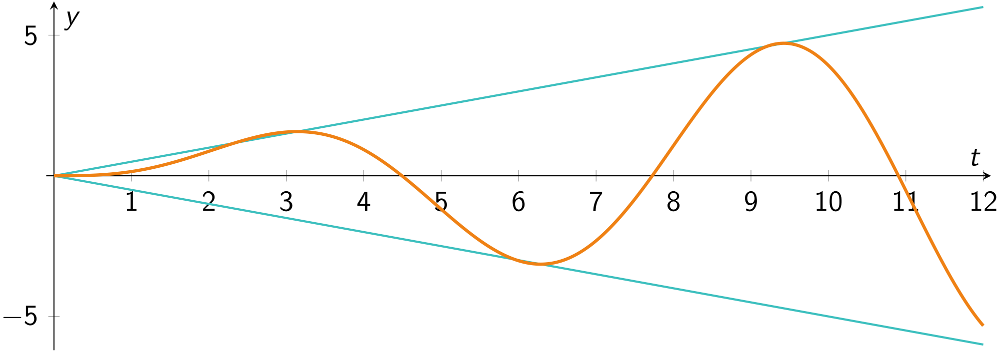
Exercise. Compute the convolution of $(f * \delta_a)(t) = \int_0^t f(\tau)\delta_a(t-\tau)\,\mathrm{d}\tau$.

Undamped Mass spring system with periodic force.

Solve the IVP

$$\begin{cases} y'' + y = \sin(t) \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

The solution is $y(t) = \frac{1}{2}\left(\sin t - t\cos t\right)$.



Tacoma Bridge (Youtube)

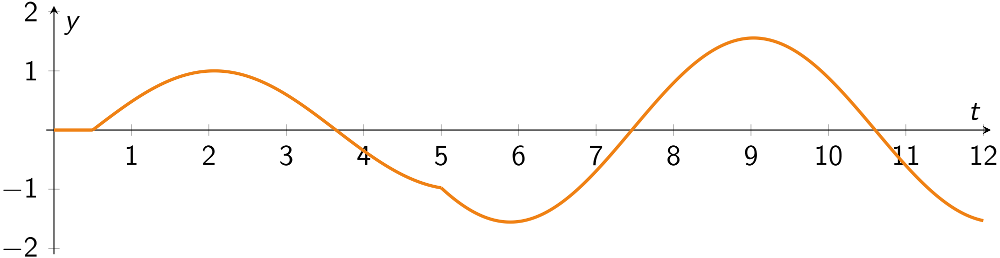
Undamped Mass spring system with two “bumps”.

Solve the IVP

$$\begin{cases} y'' + y = \delta(t-a) - \delta(t-b) \\ y(0) = 0, \\ y'(0) = 0. \end{cases}$$

The solution is $y(t) = \sin(t-a)u(t-a) - \sin(t-b)u(t-b)$.

For example with $a = \frac{1}{2}, b = 5$



Systems of ODEs

Consider the first order linear system with constant coefficients $a_{11}, a_{12}, a_{21}, a_{22}$ and known functions g_1, g_2.
Then a Systems of ODEs is given by the following IVP

$$\begin{cases} y_1'(t) = a_{11}y_1(t) + a_{12}y_2(t) + g_1(t) \\ y_2'(t) = a_{21}y_1(t) + a_{22}y_2(t) + g_2(t) \\ y_1(0) = K_1 \\ y_2(0) = K_2 \end{cases}$$