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Norwegian University of Science and Technology

TMA4125 Matematikk 4N

Numerical methods for ordinary differential equations — Runge–Kutta methods and adaptive stepping

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Introduction

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Questions:

- Can we generalise this?
- Can we do better than $\mathcal{O}(h^2)$?

Runge–Kutta methods

RK methods are one-step methods following the general scheme

$$\boldsymbol{k}_i = \boldsymbol{f}\Big(t_n + c_i h,\, \boldsymbol{y}_n + h \sum_{j=1}^{s} a_{ij} \boldsymbol{k}_j\Big), \quad i = 1, \dots, s \quad (s \text{ stages})$$

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The coefficients are all real numbers, presented in the Butcher tableau:

$$\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
c_2 & a_{21} & a_{22} & \cdots & a_{2s} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
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(????) method:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

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Heun (3rd-order):

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 \\ 2/3 & 0 & 2/3 & 0 \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

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Consistency

$$
\begin{array}{c|cccc}
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Theorem

An RK method is order-p consistent if, and only if all the conditions up to p in the table are satisfied

Consistency

p	Conditions
1	$\sum_{i=1}^{s} b_i = 1$
2	$\sum_{i=1}^{s} b_i c_i = \frac{1}{2}$
3	$\sum_{i=1}^{s} b_i {c_i}^2 = \frac{1}{3}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} c_j = \frac{1}{6}$
4	$\sum_{i=1}^{s} b_i {c_i}^3 = \frac{1}{4}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i c_i a_{ij} c_j = \frac{1}{8}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} b_i a_{ij} {c_j}^2 = \frac{1}{12}$
	$\sum_{i=1}^{s} \sum_{j=1}^{s} \sum_{k=1}^{s} b_i a_{ij} a_{jk} c_k = \frac{1}{24}$

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Error estimation

One-step methods: $\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h\boldsymbol{\Phi}(t_n, \boldsymbol{y}_n, h)$

Question: for a method $\boldsymbol{\Phi}$, how can we estimate the error $\epsilon_n = |\boldsymbol{y}_n - \boldsymbol{y}(t_n)|$, if we don't know the exact solution $\boldsymbol{y}(t)$?

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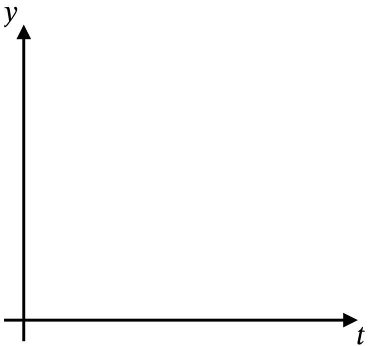
Idea

Why not use a more accurate method $\hat{\boldsymbol{\Phi}}$ to compute $\hat{\epsilon}_n = |\boldsymbol{y}_n - \hat{\boldsymbol{y}}_n|$?



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$$\hat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}(t_n + h) = Ah^{p+1}$$

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Hence, we get

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$$\hat{\epsilon}_{n+1} = \left|\hat{\boldsymbol{y}}_{n+1} - \boldsymbol{y}_{n+1}\right| \approx |A + B| h^{p+1}$$

Time-step control

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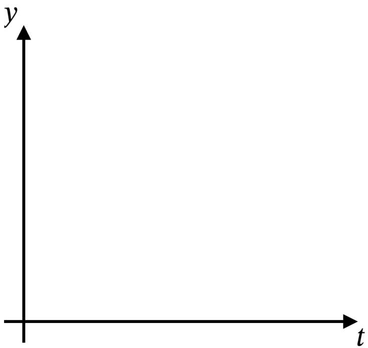

Time-step control: Heun + Euler

$$y' = 2 + 2^{-t} - \sin(0.25\pi y), \quad y(0) = 2, \quad h = 0.5, \quad \text{tol} = 0.01$$

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First step:

- Evaluate $k_1 = f(t_n, y_n) = 2$
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- Evaluate $k_2 = f(t_n + h, y_n + hk_1) = 2$
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$\Rightarrow$ No need to recompute step!

$\Rightarrow$ We can keep h as it is,
or even increase it a bit...

Time-step control: Heun + Euler

$$y' = 2 + 2^{-t} - \sin(0.25\pi y), \quad y_1 = 3, \quad h = 0.5, \quad \text{tol} = 0.01$$

Second step:

- Evaluate $k_1 = f(t_n, y_n) = 2$
- Compute Euler step: $y_{n+1}^{\text{Euler}} = y_n + hk_1 = 4$
- Evaluate $k_2 = f(t_n + h, y_n + hk_1) = 2.5$
- Compute $y_{n+1} = y_n + h\left[\frac{k_1}{2} + \frac{k_2}{2}\right] = 4.125$

Time-step control: Heun + Euler

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Error estimate: $\hat{\epsilon}_{n+1} = |4.125 - 4| = 0.125 > \text{tol}$

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Error estimate: $\hat{\epsilon}_{n+1} = |4.125 - 4| = 0.125 > \text{tol}$

$\Rightarrow$ Reduce h and recompute the step:

$$h_{new} < \left[\left(\frac{\text{tol}}{\hat{\epsilon}_{n+1}}\right)^{\frac{1}{p+1}}\right] h = \left(\frac{0.01}{0.125}\right)^{\frac{1}{2}} \times 0.5 \approx 0.14$$

Time-step control: embedded RK

We can combine a higher-order method and a lower-order method as:

$$
\begin{array}{c|cccc}
c_1 & a_{11} & a_{12} & \cdots & a_{1s} \\
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$$\boldsymbol{y}_{n+1} = \boldsymbol{y}_n + h \sum_{i=1}^{s} b_i \boldsymbol{k}_i \,, \quad \mathcal{O}(h^{p+1})$$

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Therefore, we get simply $\hat{\epsilon}_{n+1} = \left|\boldsymbol{y}_{n+1} - \boldsymbol{y}^*_{n+1}\right| = h\left|\sum_{i=1}^{s}(b_i - b_i^*)\boldsymbol{k}_i\right|$,

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so we can easily compute

$$h_{new} < \left[\left(\frac{\text{tol}}{\hat{\epsilon}_{n+1}}\right)^{\frac{1}{p+1}}\right] h$$