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Norwegian University of Science and Technology

TMA4125 Matematikk 4N

Numerical methods for ordinary differential equations — Stability and implicit stepping

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March 1, 2022

Motivation: numerical instability

Let us consider the following linear equation as our model problem:

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We want to investigate how the numerical solution behaves for different time-step sizes h, using

Euler's method: $y_{n+1} = y_n + hf(t_n, y_n)$

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The implicit Euler method

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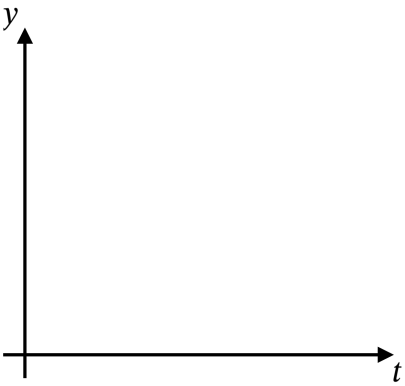

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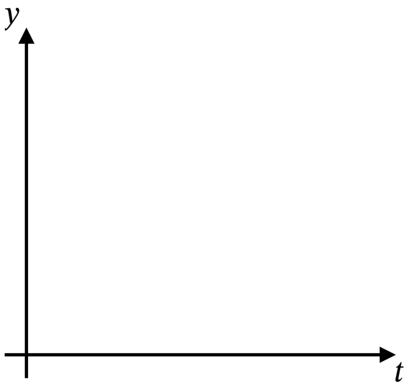

but we can also start from $y'(t_{n+1}) = f(t_{n+1}, y(t_{n+1}))$

The implicit Euler method: stability

$$y'(t) = -ky(t), \quad y(0) = y_0$$

Algorithm: $y_{n+1} = y_n + hf(t_n + h, y_{n+1})$

The implicit Euler method: stability

Stability region

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- Explicit Euler: $y_{n+1} = (1 + \lambda h) y_n$

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- Explicit Euler: $y_{n+1} = (1 + \lambda h) y_n$
- Implicit Euler: $y_{n+1} = \frac{1}{(1-\lambda h)} y_n$

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Heun's method: stability

$$y'(t) = \lambda y(t), \quad y(0) = y_0$$

Algorithm: $y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n))\right]$

Heun’s method: stability

Exercise: linear scalar ODE

$$y'(t) = -4y(t), \quad y(0) = 1 \quad \Rightarrow \quad y(t) = \mathrm{e}^{-4t}$$

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Stability intervals

- Euler: $0 < h < \frac{2}{|-4|} = \frac{1}{2}$
- Implicit Euler: $h > 0$
- Heun: $0 < h < \frac{2}{|-4|} = \frac{1}{2}$

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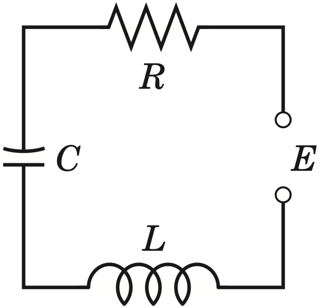
The trapezoidal method

$$\boldsymbol{y}'(t) = \boldsymbol{f}(t, \boldsymbol{y}(t))$$

Stability in linear ODE systems

$$i'' + \frac{R}{L} i' + \frac{1}{LC} i = 0, \quad i(0) = i_0$$



Source: Kreyszig, p. 3, cf. also Sec. 2.9 (p. 93)

Stability in linear ODE systems

Exercise: linear ODE system

$$R = 2,\ L = 1,\ C = 1/3 \quad \Rightarrow \quad \boldsymbol{y}'(t) = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix} \boldsymbol{y}(t)$$

Eigenvalues: $\lambda_1 = -1$, $\lambda_2 = -2$

Implicit stepping: linear ODE systems

$$\boldsymbol{y}'(t) = A\boldsymbol{y}(t) + \boldsymbol{b}(t)$$

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Implicit stepping: nonlinear ODEs

$$y'(t) = 2^{t+y}, \quad y(0) = 1, \quad h = 0.1$$

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