Theorem Let f be a 2π-periodic, square-integrable function that is Lipschitz with constant M. Then for any $x \in [-\pi,\pi)$ we have

~~Then~~ $$S_N f(x) = \sum_{k=-N}^{N} c_k(f) e^{ikx} \longrightarrow f(x) \quad \text{for } N \to \infty$$

Proof Goal. For some fixed $x \in [-\pi,\pi]$ we want to proof $f(x) - S_N f(x) \to 0$ as $N \to \infty$

We have

$$f(x) - S_N f(x) = f(x) - f * \tilde{D}_N(x) = f(x) - \int_{-\pi}^{\pi} f(y)\tilde{D}_N(x-y)\,dy \overset{\text{Symmetry}}{=} f(x) - \int_{-\pi}^{\pi} f(x-y)\tilde{D}_N(y)\,dy$$

Now we use that $\int_{-\pi}^{\pi} \tilde{D}_N(y)\,dy = 1$ multiply our first term with it and get

$$f(x) - S_N f(x) = \int_{-\pi}^{\pi} f(x)\tilde{D}_N(y)\,dy - \int_{-\pi}^{\pi} f(x-y)\tilde{D}_N(y)\,dy = \int_{-\pi}^{\pi} (f(x)-f(x-y))\tilde{D}_N(y)\,dy$$

moved inside since independent of y

Now we can plug in the formula for the normalized Dirichlet Kernel

$$f(x) - S_N f(x) = \int_{-\pi}^{\pi} \underbrace{(f(x)-f(x-y)) \cdot \frac{1}{2\pi} \cdot \frac{1}{\sin(\frac{y}{2})}}_{=g(y)} \sin((N+\tfrac{1}{2})y)$$

The function $g(y) = \dfrac{f(x)-f(x-y)}{2\pi\sin(\frac{y}{2})}$ is "nice" away from the origin, ~~but~~ where $\sin(\frac{y}{2})$ is not zero.

But for $y \to 0$ $\sin(\frac{y}{2}) \to 0$ so $\dfrac{1}{2\pi\sin(\frac{y}{2})} \to \infty$

Luckily we can use that f is Lipschitz and obtain

$$2\pi|g(y)| = \frac{|f(x)-f(y-x)|}{|\sin(\frac{y}{2})|} \leq \frac{M|y|}{|\sin(\frac{y}{2})|} \leq C$$

for some constant C, since we know (using L'Hospital's rule)

$$\lim_{y\to 0} \frac{y}{\sin(\frac{y}{2})} = \lim_{y \to 0} \frac{1}{\frac{1}{2}\cos(\frac{y}{2})} = 2$$

We proceed applying $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$ to the remaining sine:

$$f(x) - S_N f(x) = \int_{-\pi}^{\pi} g(y)\sin((N+\tfrac{1}{2})y\,dy = \overbrace{\int_{-\pi}^{\pi} g(y)\cos(\tfrac{y}{2})\sin(Ny)\,dy}^{\mathrm{I}} + \underbrace{\int_{-\pi}^{\pi} g(y)\sin(\tfrac{y}{2})\cos(Ny)\,dy}_{\mathrm{II}}$$

and we introduce $g_1(y) = g(y)\cos(\frac{y}{2})$

and $g_2(y) = g(y)\sin(\frac{y}{2})$

It remains to show that both Integrals I & II $\to 0$ for $N \to \infty$

We see that both Integrals are the real and imaginary parts of Fourier coefficients, since

$$2\pi\, I = 2\pi \cdot \int_{-\pi}^{\pi} g_1(y) \sin(Ny)\, dy = \mathrm{Re}(c_N(g_1))$$ and similarly

$$2\pi\, II = \mathrm{Im}(c_N(g_2))$$

Due to the nice behaviour of g, both g_1 and g_2 are square-integrable.

$\Rightarrow$ By Bessel's inequality and Riemann-Lebesgue

$$c_n(g_1) \to 0 \quad \text{and} \quad c_n(g_2) \to 0 \quad \text{as } n \to \infty$$

and so do both integrals. $\blacksquare$