TMA4125 Matematikk 4N

Fourier Series I: Introduction and examples.

Ronny Bergmann

Department of Mathematical Sciences, NTNU.

March 7, 2022

Motivation for periodic functions

We would like to investigate periodic phenomena, for example

- alternating current
- heart beat
- water waves

or in general phenomena that repeat.

This might also happen for complex-valued functions $f \colon \mathbb{R} \to \mathbb{C}$.

History – Jean-Baptiste Fourier & trigonometric series

Jean-Baptiste Joseph Fourier (1768 – 1830)

- First trigonometric series: Euler (1750), D. Bernoulli (1753)
- Fourier: Propagation of Heat in Solid Bodies
- He postulated

  "Every periodic function can be written as a superposition of trigonometric functions"
- rejected 1807, revised 1811, published book 1820

Periodic functions

Definition. A function $f\colon \mathbb{R} \to \mathbb{C}$ is called periodic if for some $T > 0$ it holds

$$f(x) = f(x+T) \quad \text{holds for all } x \in \mathbb{R}.$$

The value T is called the period of f.

The smallest T fulfilling the property above is called the fundamental period of f.

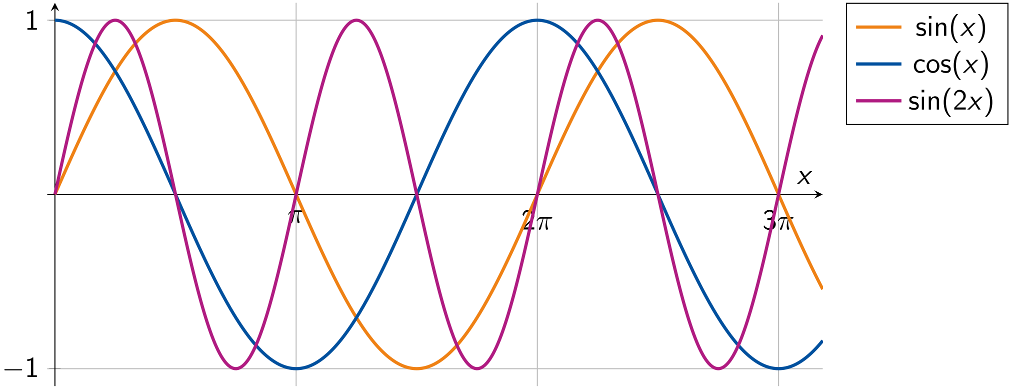
If a function g is $\frac{T}{n}$ periodic for some $n \in \mathbb{N}\backslash\{0\}$, then the number n is also called frequency of g.

- Is g T-periodic?
- What does g do in one intervall of length T, e. g. $[0, T]$?

We will usually consider functions with fundamental period $T = 2\pi$.

Periodic functions – Examples.

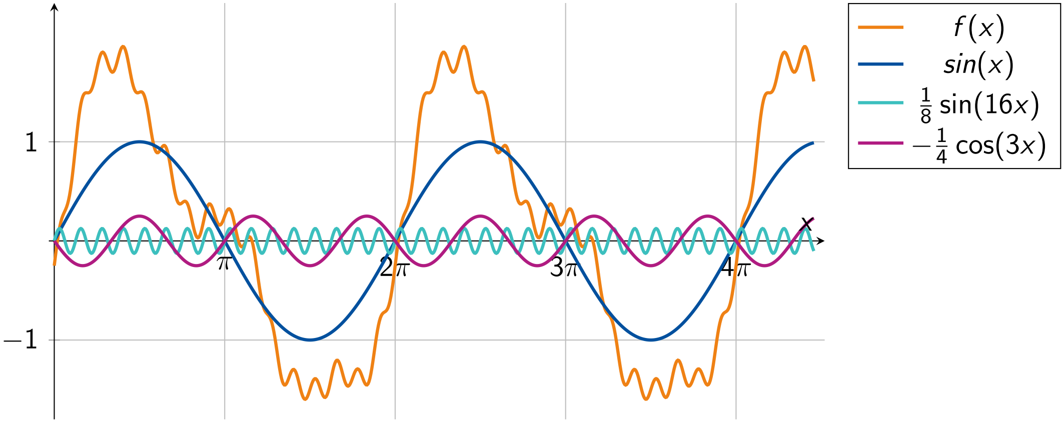
Examples. What fundamental periods do the following functions have?



What about $f(x) = \sin(nx)$ for some $n \in \mathbb{N}\setminus\{0\}$?
fundamental period T? frequency in $T = 2\pi$?

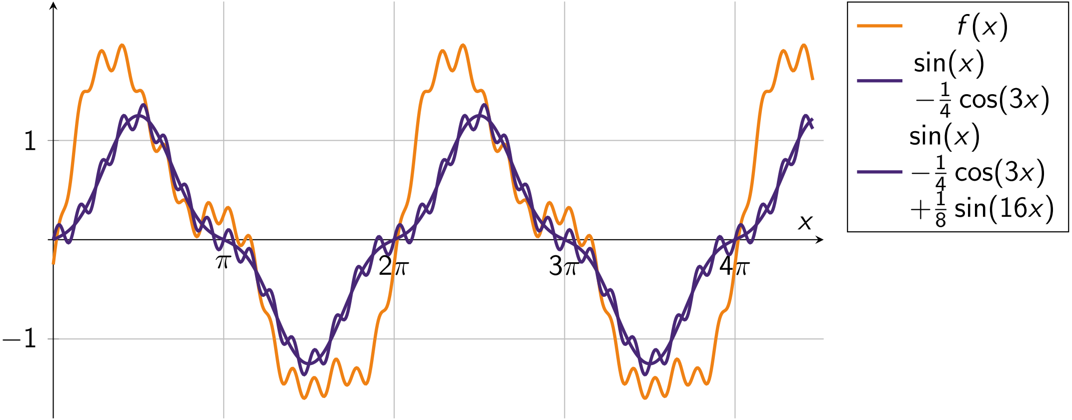
Decomposing functions

- The following function f is $T = 2\pi$ periodic and looks a little chaotic.
- Does the "coarse level look like" $\sin(x)$?.
- Does the "fine level look like it wiggles" like $\sin(16x)$?
- Does the "medium scale look like" $\cos(3x)$?
- What about "medium & coarse" combined? All 3 combined?



Decomposing functions

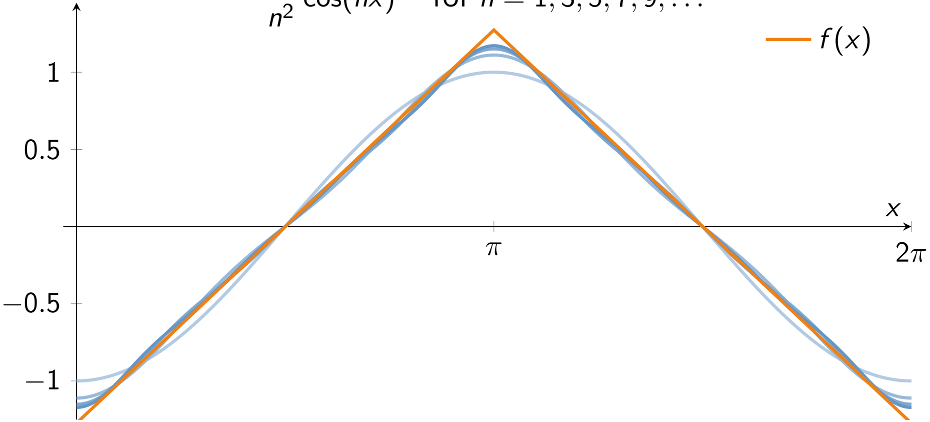
- The following function f is $T = 2\pi$ periodic and looks a little chaotic.
- Does the "coarse level look like" $\sin(x)$?.
- Does the "fine level look like it wiggles" like $\sin(16x)$?
- Does the "medium scale look like" $\cos(3x)$?
- What about "medium & coarse" combined? All 3 combined?



Composing functions

Given two functions f, g that are T periodic and $\alpha, \beta \in \mathbb{R}$.
Then $h(x) = \alpha f(x) + \beta g(x)$ is also T periodic.
$\Rightarrow$ Compose functions, for example add

$$-\frac{1}{n^2}\cos(nx) \quad \text{for } n = 1, 3, 5, 7, 9, \ldots$$



General construction scheme

Idea. use $\sin(nx)$ and $\cos(nx)$ for $n \in \mathbb{N}$ and $\cos(0x) = 1$ "to construct functions".

For coefficients a_0, a_n, b_n, $n = 1, 2, \ldots$ (real or complex), we call the function

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx).$$

a **Fourier Series**.

and we denote the Fourier partial sum f_N, $N \in \mathbb{N}$, by

$$S_N f(x) = f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx).$$

Question. What happens in the limit $\lim\limits_{N\to\infty} f_N(x)$?

Reminder. Vector space (cf. very first lecture)

A complex vector space is a set V together with operations $+$ (addition) and $\cdot$ (multiplication with a scalar) that satisfy

1. $x + y \in V$ for all $x, y \in V$
2. $x + y = y + x$ for all $x, y \in V$
3. $x + (y + z) = (x + y) + z$ for all $x, y, z \in V$
4. There exists some element $0 \in V$ such that $x + 0 = x$ for all $x \in V$
5. For all $x \in V$, there exists some element $(-x) \in V$ s. t. $x + (-x) = 0$
6. $\alpha \cdot x \in V$ for all $x \in V$ and $\alpha \in \mathbb{C}$
7. $\alpha \cdot (\beta \cdot x) = (\alpha\beta) \cdot x$ for all $x \in V$ and $\alpha, \beta \in \mathbb{C}$
8. $1 \cdot x = x$ for all $x \in V$
9. $\alpha \cdot (x + y) = \alpha \cdot x + \alpha \cdot y$ for all $x, y \in V$ and $\alpha \in \mathbb{C}$
10. $(\alpha + \beta) \cdot x = \alpha \cdot x + \beta \cdot x$ for all $x \in V$ and $\alpha, \beta \in \mathbb{C}$

NTNU

Reminder. Inner product

Let V be a complex vector space. Then a mapping $\langle\cdot,\cdot\rangle\colon V \times V \to \mathbb{C}$ is called an inner product if the following properties hold

1. Linearity in the first argument: for all $f, g, h \in V$, $\alpha, \beta \in \mathbb{C}$
$$\langle \alpha f + \beta g, h\rangle = \alpha\langle f, h\rangle + \beta\langle g, h\rangle$$
2. Conjugate symmetry: for $f, g \in V$ we have
$$\langle f, g\rangle = \overline{\langle g, f\rangle}$$
3. Positive definite: for all $f \in V$ it holds
$$\langle f, f\rangle \geq 0 \quad \text{and} \quad \langle f, f\rangle = 0 \Leftrightarrow f = 0$$

Example. Let V be the space of complex continuous 2π periodic functions. Then
$$\langle f, g\rangle = \int_{-\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$$
is a complex inner product on V.

Reminder. Euler formula

- Euler's formula states $\mathrm{e}^{\mathrm{i}x} = \cos(x) + \mathrm{i}\sin(x)$
- plugging in $-x$ yields $\mathrm{e}^{-\mathrm{i}x} = \cos(x) - \mathrm{i}\sin(x)$
- Adding both yields

$$\cos(x) = \frac{\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x}}{2}$$

- Subtracting both yields

$$\sin(x) = \frac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2i}$$

- and we also have e. g.

$$\cos(nx) = \frac{\mathrm{e}^{\mathrm{i}nx} + \mathrm{e}^{-\mathrm{i}nx}}{2} = \frac{(\mathrm{e}^{\mathrm{i}x})^n + (\mathrm{e}^{-\mathrm{i}x})^n}{2}$$

$\Rightarrow$ these are hence also called trigonometric polynomials

- But keep in mind the representation as $\mathrm{e}^{\mathrm{i}x}$ is a complex function.

The space of trigonometric functions up to degree N.

We denote by

$$V_N := \left\{ g \;\middle|\; g = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx), \quad a_0, a_1, \dots, a_N, b_1, \dots, b_N \in \mathbb{C} \right\}$$

the (real) vector space of periodic functions of degree at most N (or functions that can be “composed” by sine and cosine up to frequency N)

With the equations from last slide we can also consider

$$\tilde{V}_N := \left\{ g \;\middle|\; g = \sum_{k=-N}^{N} c_k \mathrm{e}^{\mathrm{i}kx}, \quad c_{-N}, c_{-N+1}, \dots, c_N \in \mathbb{C} \right\}$$

the complex vector space of trigonometric polynomials of degree at most N.

...and see directly that $V_N = \tilde{V}_N$.

A norm on a complex vector space

Observation. As for a real vector space, the complex inner product introduces a norm on V given by

$$\|f\| = \sqrt{\langle f, f \rangle}$$

Remember that we have the Cauchy-Schwarz inequality

$$|\langle f, g \rangle| \leq \|f\|\|g\|$$

Question. When/how can we "describe" a 2π-periodic function f "best possible" by a function $g \in V_N$, i. e. such that

$$\|g - f\| \quad \text{is as small as possible (among all } g \in V_N\text{)?}$$

Orthogonal and orthonormal systems

A sequence/family $\{\varphi_n\}_n$ is called

- orthogonal if
$$\langle \varphi_k, \varphi_n \rangle = \begin{cases} 0 & \text{if } k \neq n \\ d_n \neq 0 & \text{if } k = n, \end{cases}$$
- orthonormal if
$$\langle \varphi_k, \varphi_n \rangle = \begin{cases} 0 & \text{if } k \neq n \\ 1 & \text{if } k = n, \end{cases}$$

Note. All functions φ_k are normed since $\|\varphi_k\| = \sqrt{\langle \varphi_k, \varphi_k \rangle} = 1$

$\Rightarrow$ We can create an orthonormal system from an orthogonal one by rescaling $\tilde{\varphi}_k = \frac{1}{\sqrt{d_k}} \varphi_k$

The trigonometric system is orthonormal

Remember. for (complex) 2π periodic functions: $\langle f, g \rangle = \int_{\pi}^{\pi} f(x)\overline{g(x)}\,\mathrm{d}x$.

Example. $\varphi_k = \mathrm{e}^{\mathrm{i}kx}$, $k \in \mathbb{Z}$, is an orthogonal system.

The trigonometric system is orthonormal II

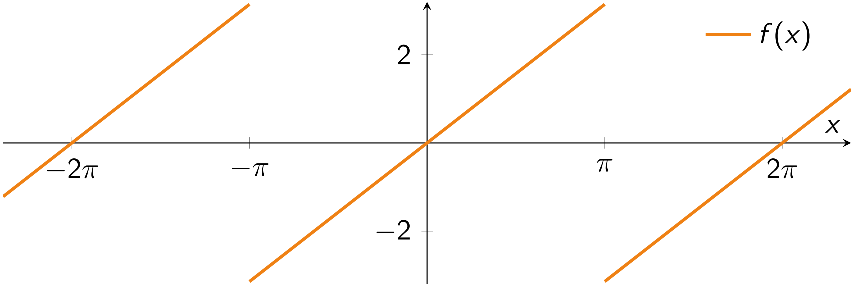
Example. We consider the family $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \ldots\}$ or shorter $\{\sin(kx)\}_{k=1}^{\infty} \cup \{\cos(kx)\}_{k=0}^{\infty}$

How to "best describe" f using V_N?

Let f be a 2π-periodic function, which function is "closest" to f from within V_N, i. e. which function $g \in V_N$ minimises $\|g - f\|$?

Example. We can define a 2π periodic function f by just defining it on one interval of length 2π and continue the rest:

$$f(x) = x, \quad \text{for } x \in [-\pi, \pi) \text{ and periodically continued.}$$



...and how to best compute g (i. e. coefficients a_k, b_k or the c_ks)?

The simpler case – the coefficients of g when f is in V_N

Assume. For the remainder let V be a complex vector space with inner product $\langle \cdot, \cdot \rangle$ and an orthogonal system $\{\varphi_k\}_k \subset V$.

$\Rightarrow V_N := \left\{ g = \sum_{k=1}^{N} d_k \varphi_k \right\}$ is a N-dimensional subspace spanned by the first N members of $\{\varphi_k\}_k$.

Proposition. A function $f \in V_N$ is of the form

$$f(x) = \sum_{k=1}^{N} c_k \varphi_k.$$

And we can compute the coefficients by

$$c_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}.$$

Even shorter. If $\{\varphi_k\}_k$ is an orthonormal system, then $c_k = \langle f, \varphi_k \rangle$.

Proof of the simpler case.

(General) Fourier coefficients

Definition. For an orthogonal system $\{\varphi_k\}_k \subset V$ and a given vector/function $f \in V$ the coefficients

$$c_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}.$$

are called (general) **Fourier coefficients**.

Sometimes we write $\hat{f}(k) = c_k$

Orthogonal Projection

For $f \in V$ we define the **projection** $\Pi_N f \in V_N$ by

$$\Pi_N f = \sum_{k=1}^{N} c_k \varphi_k \quad \text{with} \quad c_k = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}.$$

Observations.

- Π_N is a linear mapping from $f \in V$ to $\Pi_N f \in V_N$
- Π_N is indeed a projection, that is $\Pi_N f = f$ if f is already in V_N
- even more: $\Pi_N f$ is orthogonal in the sense that for any $g \in V_N$ we have

$$\left\langle f - \Pi_N f, g \right\rangle = 0$$

Best approximation theorem

Theorem. Let $f \in V$ be given. Then $\Pi_N f \in V_N$ satisfies

$$\|f - \Pi_N f\| = \min_{g \in V_N} \|f - g\|.$$

Proof. Exactly the same as in lecture 1.

Further properties

Proposition. It holds $\|\Pi_N f\| \leq \|f\|$.

Corollary. (Bessel's inequality)
Let $\{\varphi_k\}_k$ be an orthonormal system. Then for any $N \in \mathbb{N}$ we have

$$\sum_{k=1}^{N} |\hat{f}(k)| \leq \|f\|^2.$$

Corollary. (Riemann-Lebesque)
Let $\|f\| < \infty$ and $\{\varphi_k\}_k$ be an orthonormal family. Then

$$\lim_{k \to \infty} \hat{f}(k) = 0$$

(Back to) General Fourier Series

Definition. (General Fourier series for an orthogonal system)
Let $\{\varphi_k\}_{k=1}^{\infty}$ be an orthogonal system. Then the formal expression

$$\sum_{k=1}^{\infty} \hat{f}(k)\varphi_k, \qquad \hat{f}(k) = \frac{\langle f, \varphi_k \rangle}{\|\varphi_k\|^2}$$

is called a **general Fourier series**.

Since $\Pi_N f = \sum_{k=1}^{N} \hat{f}(k)\varphi_k$ is the partial sum, so we write $S_N f = \Pi_N f$.

For the specific case for periodic functions we still need

$$V := \left\{ f : [-\pi, \pi) \to \mathbb{C} \;\middle|\; \int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x < \infty \right\},$$

where we assume that the integral exists.
Functions $f \in V$ are said to be square-integrable.
V is indeed a vector space.

Complex trigonometric series / Fourier series

Definition. (Complex Fourier series)
Let f be a 2π-periodic function and consider the orthogonal system

$$\left\{\mathrm{e}^{\mathrm{i}kx}\right\}_{k\in\mathbb{Z}}$$

on $[-\pi, \pi)$.
Then the formal series

$$\sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx}, \quad \text{with } c_k = \frac{\langle f, \mathrm{e}^{ikx}\rangle}{\|\mathrm{e}^{\mathrm{i}kx}\|^2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

is called the **complex Fourier series** associated with f.
We also write $c_k(f)$ or $\hat{f}(k)$ for the c_k.

We write this association also as

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx}$$

Real trigonometric series / Fourier series

Definition. (Real Fourier series)
Let f be a 2π-periodic function and consider the orthogonal system

$$\{1\} \cup \{\cos(nx)\}_{n\in\mathbb{N}} \cup \{\sin(nx)\}_{n\in\mathbb{N}}$$

Then the formal series associated to f given by

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

with coefficients

$$a_0 = 2\frac{\langle f,1\rangle}{\langle 1,1\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,\mathrm{d}x, \qquad a_n = \frac{\langle f,\cos(nx)\rangle}{\|cos(nx)\|^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,\mathrm{d}x$$

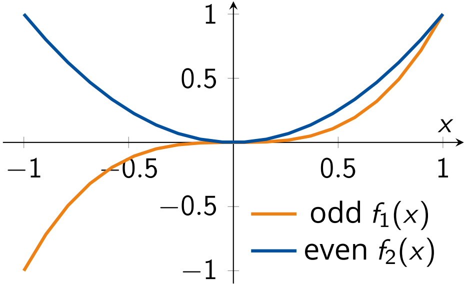
$$b_n = \frac{\langle f,\sin(nx)\rangle}{\|sin(nx)\|^2} = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,\mathrm{d}x$$

is called the **(real) Fourier series** associated with f.

Note.
a_0 and a_n,b_n, $n \in \mathbb{N}$, are all real values $\Leftrightarrow$ f is real valued.

Reminder. Even and odd functions

A function is called even if $f(x) = f(-x)$ holds for all x.
A function is called odd if $f(x) = -f(-x)$ holds for all x.



we have the following

f	g	$f \cdot g$
even	even	even
odd	even	odd
even	od	odd
odd	odd	even

Examples. For $n \in \mathbb{N}$: $\cos(nx)$, is even, $\sin(nx)$ is odd.

Even better.

- if f is odd then $\int_{-L}^{L} f(x)\,\mathrm{d}x = 0$
- if f is even then $\int_{-L}^{L} f(x)\,\mathrm{d}x = 2\int_{0}^{L} f(x)\,\mathrm{d}x$

Relations between the real and complex Fourier series

We can use Eulers formula to relate the (complex) Fourier coefficients c_k and the (real) Fourier coefficients a_0, a_n, b_n:

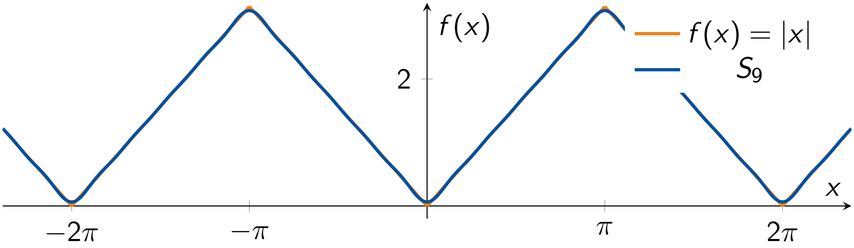
$$\begin{aligned}
a_n &= c_n + c_{-n}, && n = 0, 1, \ldots, \\
b_n &= \mathrm{i}(c_n - c_{-n}), && n = 1, 2, \ldots, \\
c_0 &= \frac{a_0}{2} \\
c_n &= \frac{a_n - ib_n}{2}, && n = 1, \ldots, \\
c_{-n} &= \frac{a_n + ib_n}{2}, && n = 1, 2, \ldots,
\end{aligned}$$

and even more: the projection onto $V_N = \tilde{V}_N$ yield the same partial sum

$$S_N(f) = \sum_{k=-N}^{N} c_k \mathrm{e}^{\mathrm{i}kx} = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx).$$

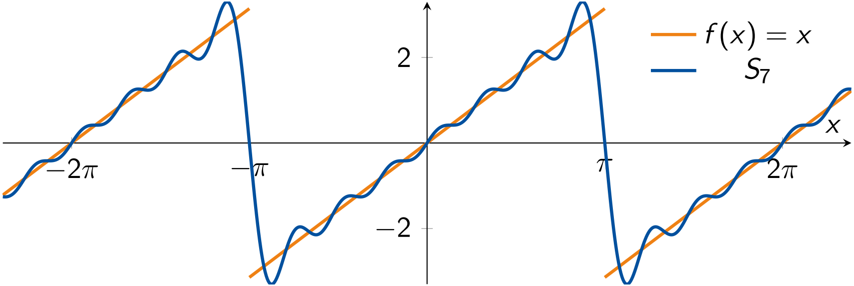
$\Rightarrow$ Choose the coefficients that "fit better".

Example 1. Fourier coefficients of $f(x) = |x|$, $x \in [-\pi, \pi)$.



Question. Does $S_N f$ tend to f for $N \to \infty$? And if so, in what sense?

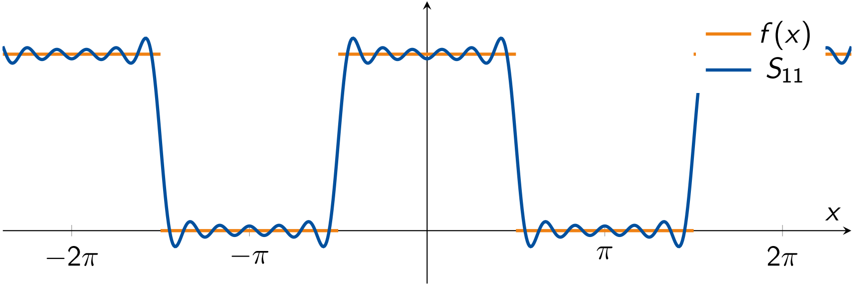
Complex Fourier coefficitents of $f(x) = x$



Here we can ask the same question about convergence as well.

(Shifted) Heaviside function

Consider the function $f(x) = \begin{cases} 1 & \text{for } |x| \leq \frac{\pi}{2} \\ 0 & \text{else.} \end{cases}$



...but let's turn to Python to really see the effect.