NTNU

Norwegian University of Science and Technology

TMA4125 Matematikk 4N

Fourier Series II: Convergence & further properties

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Convergence – Motivation

We introduced the Fourier series

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

with

$$\hat{f}(k) = \langle f, \mathrm{e}^{\mathrm{i}kx} \rangle = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

Question. When/For which x does

$$f(x) = \sum_{k=-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx} = \lim_{N\to\infty} S_N f(x)$$

hold?

Remember that $S_N f$ is the Fourier partial sum

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx}$$

The Dirichlet kernel

The Nth Dirichlet kernel is defined by

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And since $\int_{-\pi}^{\pi} \cos(nx) = 0$ for any $n = 1, 2, \ldots$ we get

$$\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = 2\pi.$$

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$$\int_{-\pi}^{\pi} D_N(x)\,\mathrm{d}x = 2\pi.$$

We can define the normalized Dirichlet Kernel

$$\tilde{D}_N(x) = \frac{1}{2\pi} D_N(x) \quad \text{where we get} \quad \int_{-\pi}^{\pi} \tilde{D}_N(x)\,\mathrm{d}x = 1$$

Rewriting the Dirichlet kernel

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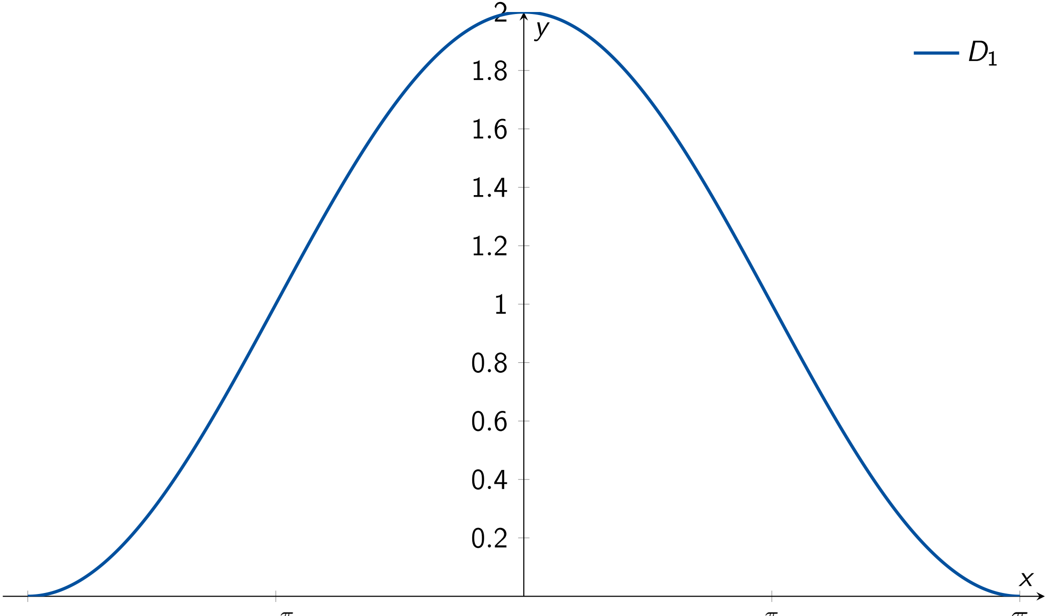
Using a shift in the sum index and the geometric sum with $q = \mathrm{e}^{\mathrm{i}x}$ we can rewrite

$$D_N(x) = \sum_{k=-N}^{N} \mathrm{e}^{\mathrm{i}kx} = \frac{\sin\left((N + \frac{1}{2})x\right)}{\sin\left(\frac{x}{2}\right)}$$

Plotting the Dirichlet Kernel



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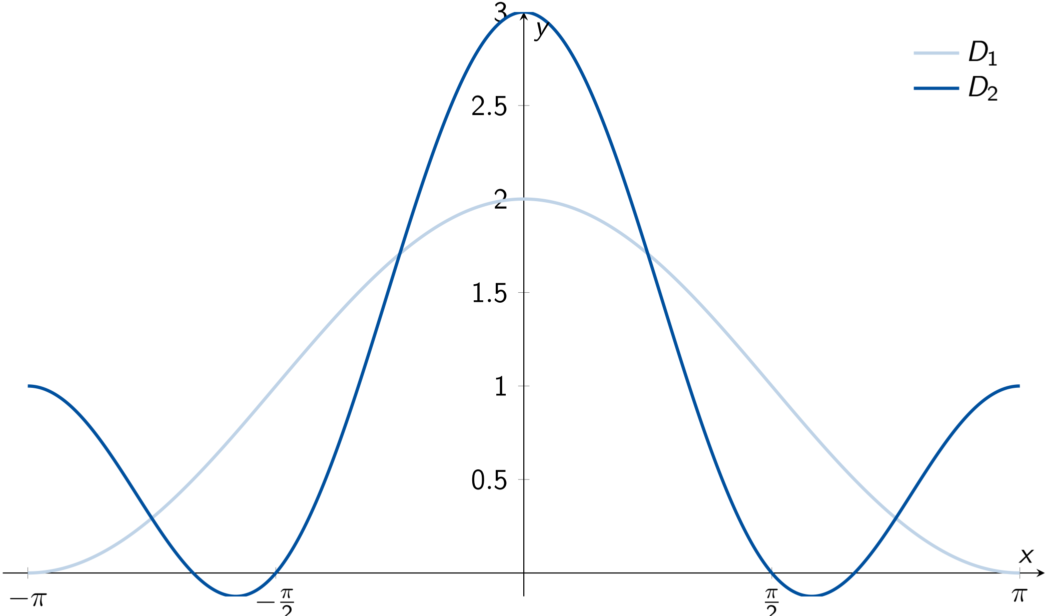

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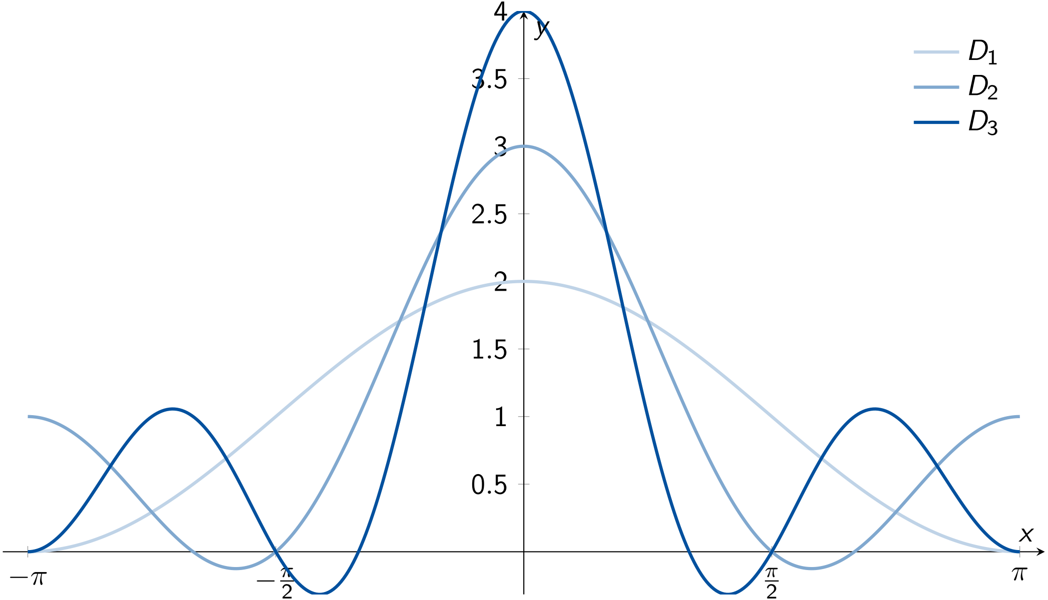

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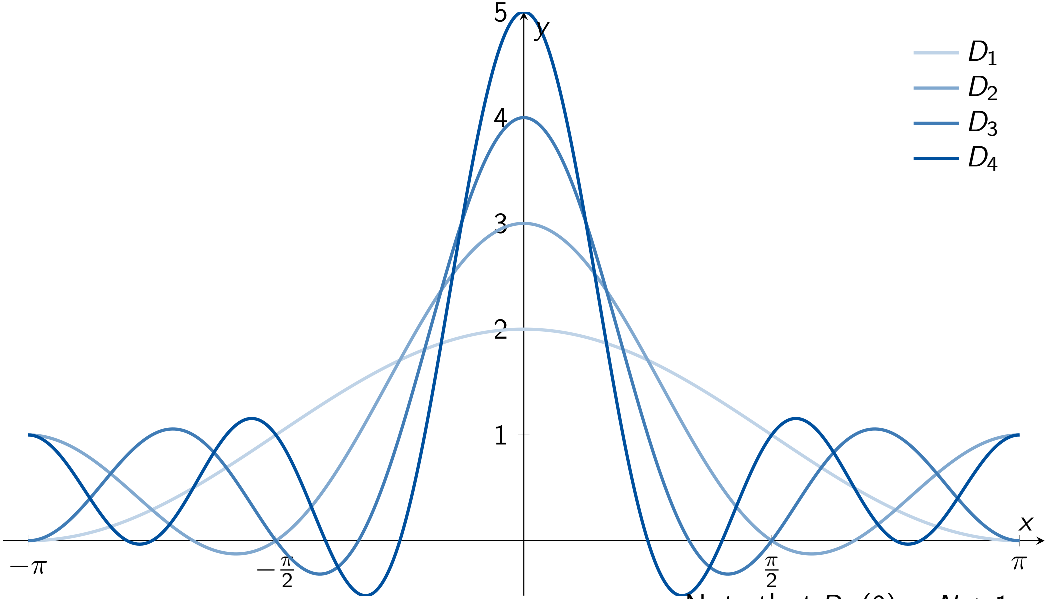

Note that $D_N(0) = N + 1$

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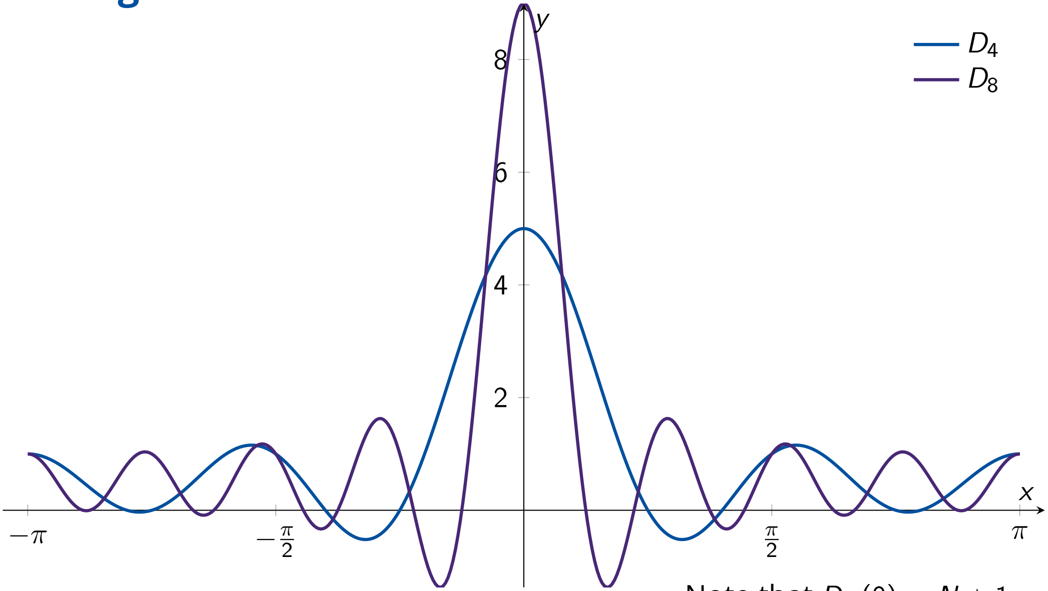

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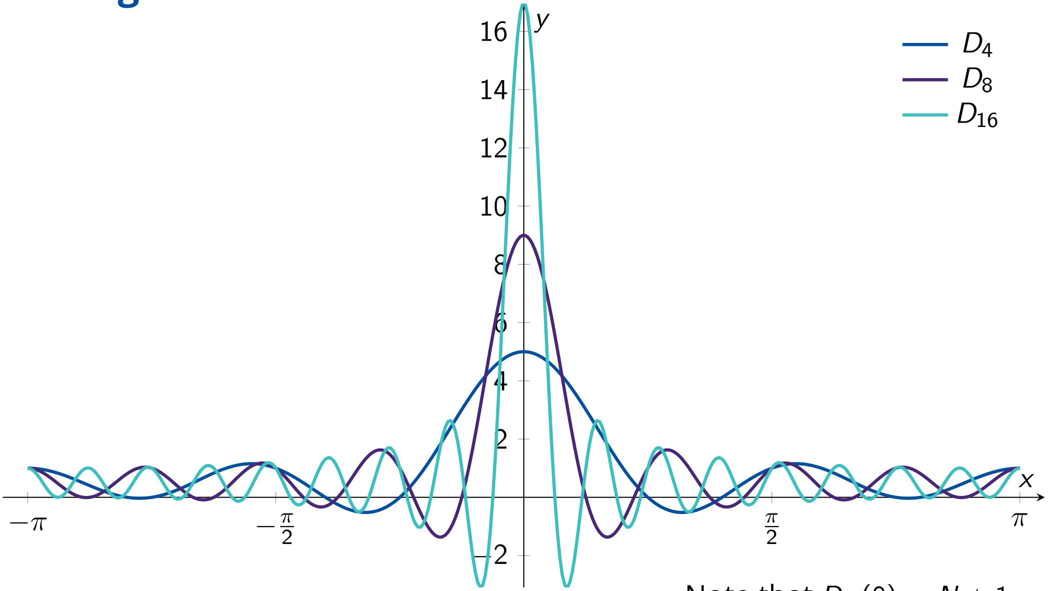

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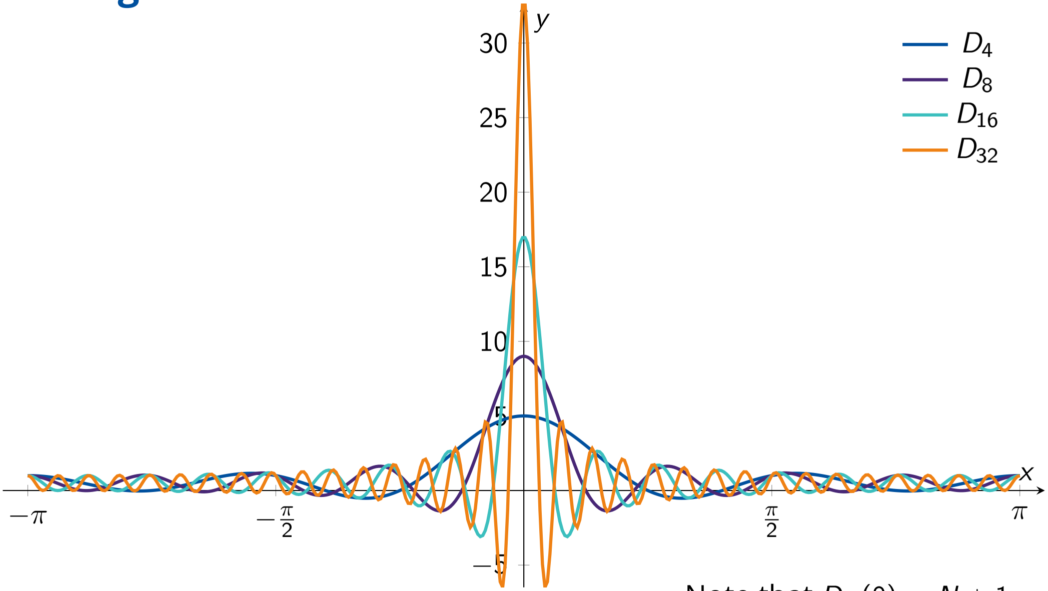

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Fourier Partial Sums and the Dirichlet Kernel

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Plugging in the Fourier coefficients, we recognise that

$$S_N f(x) = \int_{-\pi}^{\pi} f(y) \tilde{D}_N(x - y) \, \mathrm{d}y,$$

where $\tilde{D}_N$ is the normalized Dirichlet kernel. But we already saw this operation

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Recap. Convolution

Definition. For two 2π-periodic functions f, g the convolution (norsk.: konvolusjon, dt.: Faltung) is denoted by $f * g$ and defined by

$$(f * g)(x) = \int_{-\pi}^{\pi} f(y)g(x - y)\,\mathrm{d}y.$$

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Lemma. For two 2π-periodic functions it holds that

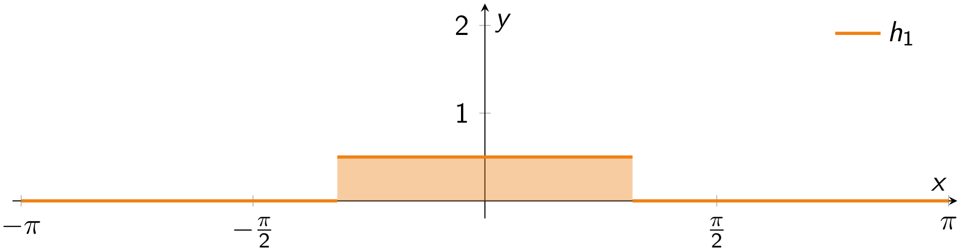
$$(f * g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)\,\mathrm{d}y. = \int_{-\pi}^{\pi} g(y)f(x-y)\,\mathrm{d}y = (g * f)(x).$$

Proof.

Understanding Convolution – Avergaging

Let's convolve our function f first with a special function to understand convolutions better.
For $0 < \varepsilon < \pi$ let h_ε be the 2π periodic function given by

$$h_\varepsilon(x) = \frac{1}{2\varepsilon}X_{[-\varepsilon,\varepsilon]}(x) = \begin{cases} \frac{1}{2\varepsilon} & \text{for } |x| \leq \varepsilon \\ 0 & \text{else.} \end{cases}$$



Note that we have (similar to the dirac idea): $\displaystyle\int_{-\pi}^{\pi} h_\varepsilon(x)\,\mathrm{d}x = 1.$

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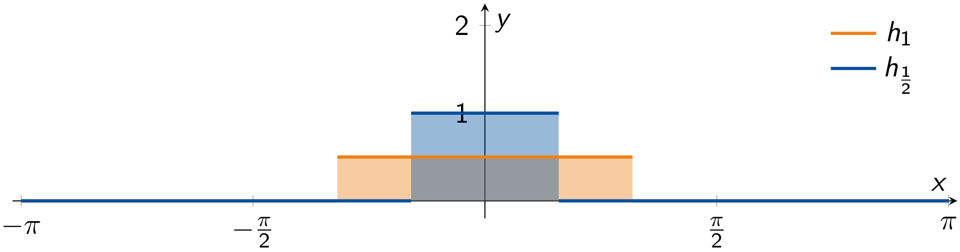

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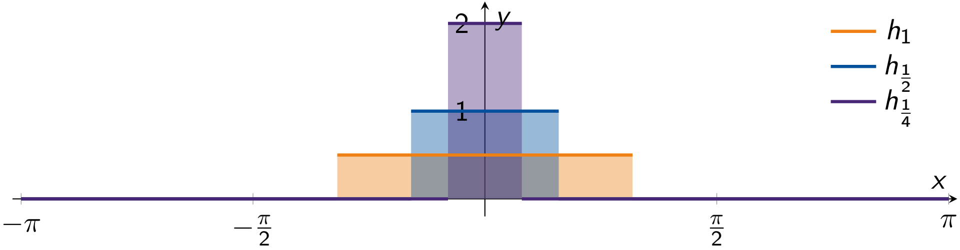

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If f is continuous this follows directly from the fundamental theorem of calculus:

Convolving with h_ε

For the general case $(f * h_\varepsilon)(x)$ let's first introduce

- mirror operator $\sigma\colon f \mapsto \sigma f$ defined by $\sigma f(y) = f(-y)$
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So $(f * h_\varepsilon)(x)$ represents a sort of average value of f taken in a (small) interval centered at x.

Similarly to the case $x = 0$ we get here for a continuous f as well that $f * h_\varepsilon(x) \to f(x)$ as $\varepsilon = 0$.

(Back to) Convergence of Fourier Series.

Remember. We saw that we can write the Fourier partial sum as a convolution

$$S_N f(x) = (f * \tilde{D}_N)(x) = \int_{-\pi}^{\pi} f(y)\tilde{D}_N(x-y)\,\mathrm{d}y,$$

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For $N \to \infty$ the “mass” of $\tilde{D}_N$ is still 1 and concentrates around 0.

So for f being “nice enough” we get $\lim_{N\to\infty} S_N f(x) = f(x)$ for any x.

Warning on $\tilde{D}_N$ and Lipschitz continuity

Warning. The Dirichlet kernel is “quite nasty”, especially since it it’s very oscillatory. So there exist functions continuous at x, where $S_N f(x)$ does **not** converge to $f(x)$!

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Remedy. A slightly stronger condition on f than (just) continuity.

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Remedy. A slightly stronger condition on f than (just) continuity.

Definition. A function f is Lipschitz continuous at x if

$$|f(x-y) - f(x)| \leq M|y| \qquad \text{for all } y \in (-\pi, \pi)$$

f is called Lipschitz (with constant M) if it is Lipschitz at all $x \in [-\pi, \pi]$.

Pointwise Convergence of Fourier Series I

Theorem. (Pointwise convergence of Fourier Series for Lipschitz continuous functions)

Let f be a 2π-periodic function which is square integrable and Lipschitz with constant M.

Then

$$S_N f(x) = \sum_{n=-N}^{N} c_k \mathrm{e}^{\mathrm{i}kx} \to f(x) \quad \text{for } N \to \infty$$

for any $x \in [-\pi, \pi)$.
This we also call pointwise convergence.

Proof. You can find the proof in an extra note in the wiki.

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Proof. You can find the proof in an extra note in the wiki.

Note. If the function is continuously differentiable, it is Lipschitz.

Pointwise Convergence of Fourier Series II

Theorem.

Let f be a 2π periodic function that is piecewise continously differentiable such that at a jump the left and right hand side limits

$$f(x^-) := \lim_{y \to x^-} f(x) \qquad \text{and } f(x^+) := \lim_{y \to x^+} f(x)$$

as well as the left and right derivatives at the jumps x

$$\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} \qquad \text{and } \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

exist.

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exist. Then the Fourier series converges to the mit point

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}\big(f(x^-)+f(x^+)\big)$$

At points where f is continuous, we obtain pointwise convergence to f

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Proof. We omit the proof here.

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L^2 convergence

Remember the space of square integrable functions

$$V = \left\{ f \colon [-\pi, \pi) \to \mathbb{C} \;\middle|\; \|f\|^2 = \int_{-\pi}^{\pi} |f(x)|^2 \,\mathrm{d}x < \infty \right\}$$

Here $\|f\|$ is also called the L^2-norm.

[1] cf. Plonka, Potts, Steidl, Tasche: Numerical Fourier Analysis, Birkhäuser, 2018, Section 1.3

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Theorem 3. Let $f \in L^2((-\pi,\pi))$. Then $S_N f = \Pi_N f$ converges to f in L^2 norm, i. e.

$$\lim_{N\to\infty} \|f - S_N f\| = 0$$

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Fourier Series on $[-L, L]$ – Idea

We can also consider functions of period $2L$, $L > 0$.
Idea. “Stretch/Squeeze” all formulae we had until now.

Example. since $\sin x$ is 2π-periodic, $\sin \frac{\pi}{L}x$ is $2L$-periodic.

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Then

$$\left\{ \mathrm{e}^{\frac{\mathrm{i}kx\pi}{L}} \right\}_{k\in\mathbb{Z}} \quad \text{and} \quad \left\{ \cos \frac{n\pi x}{L} \right\}_{n=0}^{\infty} \cup \left\{ \sin \frac{n\pi x}{L} \right\}_{n=1}^{\infty}$$

are orthogonal systems on $[-L, L]$ with respect to the scalar product

$$\langle f, g \rangle = \int_{-L}^{L} f(x)\overline{g(x)}\,\mathrm{d}x$$

Fourier Series on $[-L, L]$

The Fourier series read

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\frac{\mathrm{i}kx\pi}{L}}, \qquad c_k = \frac{1}{2L} \int_{-L}^{L} f(x) \mathrm{e}^{-\frac{\mathrm{i}kx\pi}{L}} \, \mathrm{d}x$$

and

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}$$

with coefficients

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(x) \, \mathrm{d}x, \quad a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, \mathrm{d}x, \quad b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, \mathrm{d}x.$$

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Even and odd extensions

Definition. (nearly a reminder)

A function f on a symmetric interval $I = [-L, L]$, $L > 0$, is called

- is called even if $f(x) = f(-x)$ holds for all $x \in I$.
- is called odd if $f(x) = -f(-x)$ holds for all $x \in I$.

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These mean that it is enough to know a function on half the interval and we can extend it

Definition. For $f\colon [0, L] \to \mathbb{R}$ we define its

- odd extension $f_{\mathrm{o}}(x) = \begin{cases} f(x) & \text{if } x \in [0, L) \\ -f(-x) & \text{if } x \in (-L, 0) \end{cases}$
- even extension $f_{\mathrm{e}}(x) = \begin{cases} f(x) & \text{if } x \in [0, L) \\ f(-x) & \text{if } x \in (-L, 0) \end{cases}$

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Note. Both f_o and f_e extend f from $[0, L]$ to $[-L, L]$.

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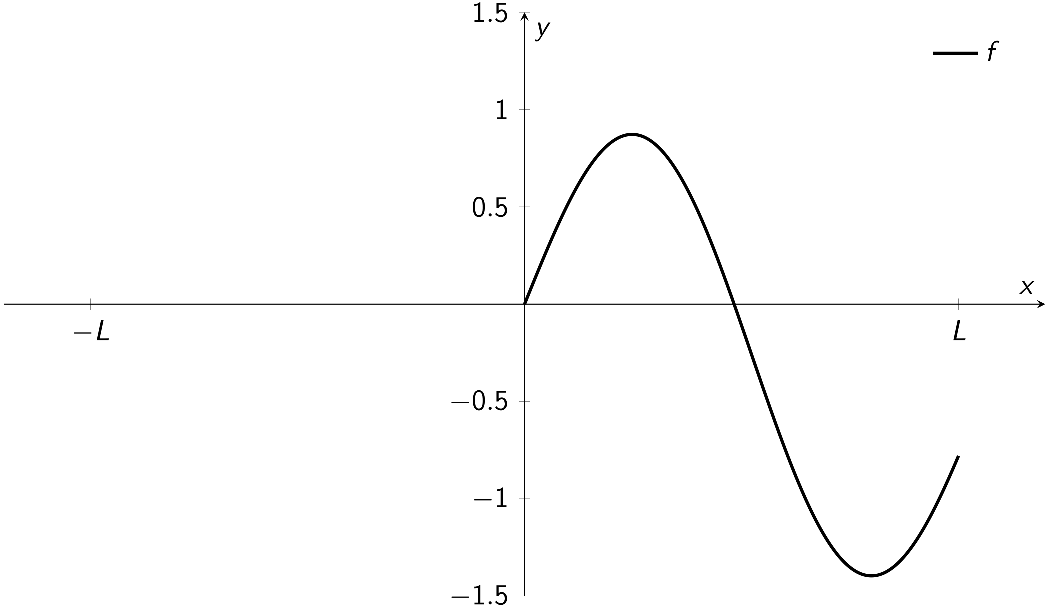
Illustration of even and odd extensions



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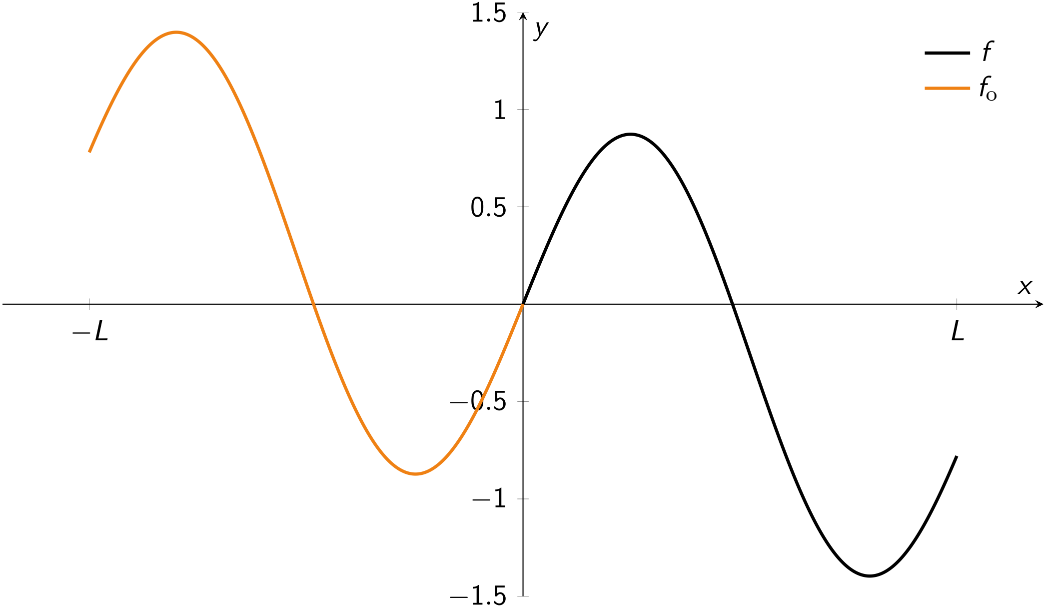

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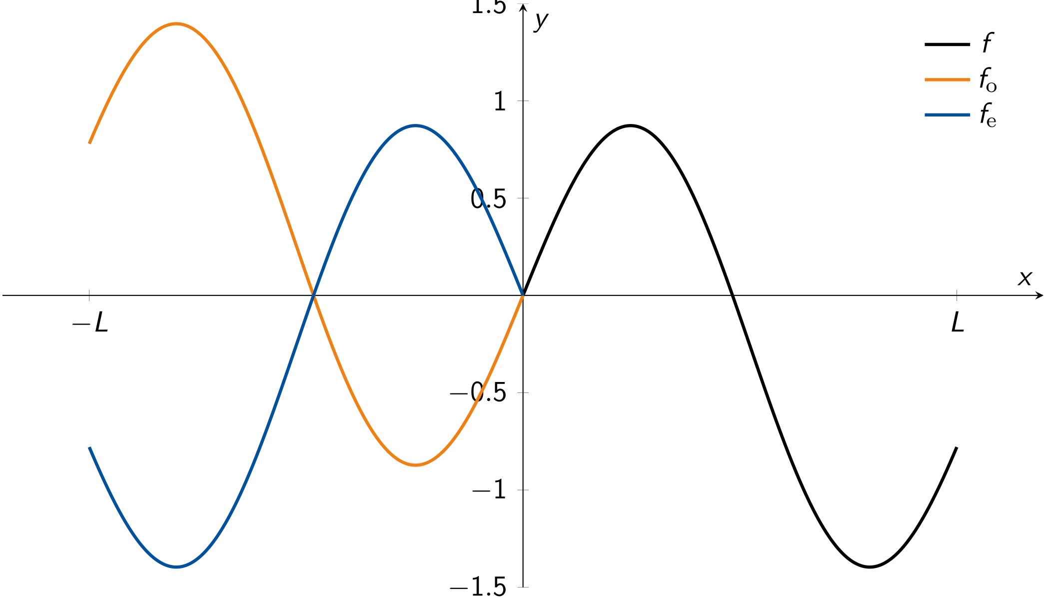

Observation for extensions

Observe that we can compute the Fourier series of f_e and f_o by just using f.

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For f_o we get

$$a_0(f_\mathrm{o}) = 0 \qquad a_n(f_\mathrm{o}) = 0 \qquad b_n(f_\mathrm{o}) = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,\mathrm{d}x$$

or in other words a sine series

Observation for extensions

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For f_{e} we get

$$a_0(f_{\mathrm{e}}) = \frac{2}{L}\int_0^L f(x)\,\mathrm{d}x \qquad a_n(f_{\mathrm{e}}) = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,\mathrm{d}x \qquad b_n(f_{\mathrm{e}}) = 0$$

or in other words a cosine series

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Parseval Identity

Theorem.

Let f be given such that $\int_{-L}^{L} |f(x)|^2 \,\mathrm{d}x$ exists and is finite (or similarly: $f \in L^2((-L, L))$). Let

$$
\begin{aligned}
f &\sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\frac{\mathrm{i}kx\pi}{L}} \\
&\sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L}
\end{aligned}
$$

Then

$$
\frac{1}{2L}\|f\|^2 = \frac{1}{2L}\int_{-L}^{L} |f(x)|^2 \,\mathrm{d}x = \sum_{k=-\infty}^{\infty} |c_k|^2 = \frac{a_0^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)
$$

Proof.

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Application of Parsevals Identity

For the heaviside function $u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ defined on $x \in [-\pi, \pi]$ we can compute its Fourier series

$$u(x) \sim \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x$$

and u is smooth in $x = \frac{\pi}{2}$ and satisfies the assumptions of the last theorem.

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Note that the sine alternates between ± 1 at the points. Thus rearranging yields

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{2}(1 - \frac{1}{2}) = \frac{\pi}{4}$$

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Spectrum of periodic functions

For a $2L$-periodic function we defined its Fourier series as

$$f \sim \sum_{k=-\infty}^{\infty} \hat{f}(k)\mathrm{e}^{\mathrm{i}kx\frac{\pi}{L}}$$

where

$$\hat{f}(k) = c_k = \frac{1}{2L}\int_{-L}^{L} f(x)\mathrm{e}^{-\mathrm{i}kx\frac{\pi}{L}}\,\mathrm{d}x \in \mathbb{C}.$$

We can associate with f a sequence of pairs $\left(c_k, \frac{k}{2L}\right)_{k\in\mathbb{Z}}$. which is called the spectrum of f.

Amplitude

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Since $\hat{f}(k) = c_k \in C$ we can also write this complex number as

$$c_k = |c_k| \mathrm{e}^{\mathrm{i}\theta_k},$$

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The number $\frac{1}{2L}$ is called the frequency.

Further properties

If we have two 2π-periodic functions f, g with associated Fourier series $\sum_{k\in\mathbb{Z}} c_k(f)\mathrm{e}^{\mathrm{i}kx}$ and $\sum_{k\in\mathbb{Z}} c_k(g)\mathrm{e}^{\mathrm{i}kx}$.

- real functions and $c_k(f)$ if f is real-valued, then $c_k(f) = \overline{c_{-k}(f)}$
- Linearity. the Fourier coefficients of $h_1(x) = \alpha f(x) + \beta g(x)$ ($\alpha, \beta \in \mathbb{R}$) are given by $c_k(h_1) = \alpha c_k(f) + \beta c_k(g)$ for $k \in \mathbb{Z}$.

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- Translation of a function: For $h_2(x) = f(x - x_0)$ for some $x_0 \in [-\pi, \pi)$ yields the Fourier coefficients are $c_k(h_2) = \mathrm{e}^{\mathrm{i}kx_0} c_k(f)$.

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- Modulation of a function: For $h_3(x) = \mathrm{e}^{-\mathrm{i}k_0} f(x)$ for some $k_0 \in \mathbb{Z}$ the Fourier coefficients are $c_k(h_3) = c_{k+k_0}(f)$.

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- Differentiation. if f is absolutely continuous and both f, f' are on L_1 (their absolute value ins integrable) then

$$c_k(f') = 2\pi i k c_k(f)$$

Addendum: Convolution and Fourier Coefficients

Using $(f * g)(x) = \int_{-\pi}^{\pi} f(y)g(x-y)$ we compute (for $k \in \mathbb{Z}$)

$$c_k(f * g) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(y)g(x-y)\,\mathrm{d}y\; \mathrm{e}^{-\mathrm{i}kx}\,\mathrm{d}x$$

We use Fubini to switch the integrals, substitute $t = x - y$ in the inner integral (w.r.t x) and we "snuck in" with $1 = \mathrm{e}^{\mathrm{i}ky}\mathrm{e}^{-\mathrm{i}ky}$

$$\begin{aligned} c_k(f * g) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\mathrm{e}^{-\mathrm{i}ky}\int_{-\pi}^{\pi} g(x-y)\mathrm{e}^{-\mathrm{i}k(x-y)}\,\mathrm{d}x\,\mathrm{d}y \\ &= 2\pi\frac{1}{2\pi}\int_{-\pi}^{\pi} f(y)\mathrm{e}^{\mathrm{i}ky}\frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\mathrm{e}^{-\mathrm{i}kt}\,\mathrm{d}t\,\mathrm{d}y \end{aligned}$$

now the inner integral is $c_k(g)$ and indepent of y, the remaining one is then just $c_k(f)$ - in summary

$$c_k(f * g) = 2\pi c_k(g)c_k(f)$$

Note. Some books define the convolution with a factor $\frac{1}{2\pi}$ upfront, then it vanishes in the last line here as well.