NTNU

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TMA4125 Matematikk 4N

Heat Equation

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Recap: Partial Differential Equations

We already discussed ODEs: Equations that include derivatives of (univariate) functions.

Definition. A partial differential equation (PDE) is an equation that involves one or more partial derivatives of an (multivariate, unknown) function u.

We often use $u(t, x)$, $u(t, x, y)$ or $u(t, x, y, z)$ for a function that depends on time (t) and space (1D, 2D, or 3D, respectively).

- the PDE is linear if it is of first degree in u and its derivatives.
- other wise it is called nonlinear.
- It is called homogenous if all terms include u or one of its partial derivatives
- otherwise it is called nonhomogeneous

Solutions of partial Differential Equations

A solution u of a PDE in some region $\Omega \subset D$ in the space of its variables t, x (y,z) is a function whose partial derivatives (appearing in the PDE) exist in D and such that u fulfills the PDE on Ω.

Similar to ODEs, the set of solutions might be huge, so for a unique solution we additionally require for example

- that u is given on the boundary of the region Ω (boundary condtions)
- that u has some conditions for the start time $t = 0$ (initial conditions)

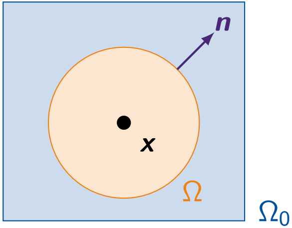
Theorem. (Superposition principle) If u_1 and u_2 are solutions of a homogeneous linear PDE on some Ω, then

$$u = au_1 + bu_2, \qquad \text{for some constants } a, b$$

is also a solution of that PDE in the region Ω.

The Heat Equation – Derivation

Goal.
Model the distribution of heat in a given area over time.



- an area (our lecture hall) Ω_0
- Consider an area Ω in our lecture hall
- At each boundary point: normal vector $\boldsymbol{n}$
- Let's look at one point $\boldsymbol{x} \in \Omega$.

We model/describe the

- density of internal energy $e(\boldsymbol{x}, t)$ (in $[J/m^3]$) at a point $\boldsymbol{x}$ at time t
- heat flux through a surface $\partial\Omega$ is a vector field $\boldsymbol{F}(\boldsymbol{x}, t)$ ($[J/(m^2 s)]$)
- power density $p(\boldsymbol{x}, t)$ (in $[J/(m^3 s)]$,
  imagine e. g. a candle

Heat Equation – Step 1: Conservation of Energy

For any $\Omega \subset \Omega_0$ the **Conservation of energy** dictates

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} e(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega} p(\boldsymbol{x},t)\,\mathrm{d}V - \int_{\partial\Omega} \boldsymbol{F}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})\,\mathrm{d}S$$

Remember. Using Gauß (divergence) theorem we can rewrite

$$\int_{\partial\Omega} \boldsymbol{F}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})\;\mathrm{d}S = \int_{\Omega} \nabla\cdot\boldsymbol{F}(\boldsymbol{x},t)\,\mathrm{d}V$$

Plugging this into our conservation of energy we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega} e(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega} \frac{\partial}{\partial t} e(\boldsymbol{x},t)\,\mathrm{d}V = \int_{\Omega} \Big(p(\boldsymbol{x},t) - \nabla\cdot\boldsymbol{F}(\boldsymbol{x},t)\Big)\,\mathrm{d}V$$

Since this has to hold for any $\Omega \subset \Omega_0$ we obtain

$$\frac{\partial}{\partial t} e(\boldsymbol{x},t) + \nabla\cdot\boldsymbol{F}(\boldsymbol{x},t) = \rho(\boldsymbol{x},t) \qquad \text{for all } \boldsymbol{x}\in\Omega_0 \text{ and } t > 0.$$

Heat Equation – Step 2: Constitutive Laws

Problem. We can not derive the internal energy and we are also only interested in the (absolute) temperature $T = T(\boldsymbol{x}, t)$(e. g. in $[K]$).

But we can use two empirical observations.

The first constitutive law relates $e(\boldsymbol{x}, t)$ to T:

$$e = e_0 + \sigma(T - T_0) = e_0 + \sigma\vartheta, \qquad \text{where}$$

- e_0 is the base energy density related to T_0 (e. g. $0^\circ K$)
- $\sigma = \sigma(\boldsymbol{x})$ ($[J/(m^3K)]$) the specific heat capacity (at $\boldsymbol{x}$)
- $\vartheta := T - T_0$ relative temperature (we are actually interested in).

The second constitutive law (Fourier's law) relates T (or ϑ) to $\boldsymbol{F}$

$$\boldsymbol{F} = -\lambda\nabla\vartheta,$$

where $\lambda = \lambda(\boldsymbol{x})$ ($[J/(mKs)]$) is the heat conductivity.

The Heat Equation (finally!)

Plugging both constitutive laws ($e = e_0 + \sigma\vartheta$ and $\boldsymbol{F} = -\lambda\nabla\vartheta$) into the energy conservation yields the

heat equation.

$$\sigma\frac{\partial}{\partial t}\vartheta - \nabla\cdot(\lambda\nabla\vartheta) = p \qquad \text{for all } \boldsymbol{x} \in \Omega_0 \text{ and } t > 0$$

If we assume that λ does not depend on $\boldsymbol{x}$ (e. g. a homogeneous material) we obtain

$$\sigma\frac{\partial}{\partial t}\vartheta - \lambda\Delta\vartheta = p \qquad \text{for all } \boldsymbol{x} \in \Omega_0 \text{ and } t > 0$$

Or if we divide by σ and introduce $c^2 = \frac{\lambda}{\sigma} > 0$

$$\frac{\partial}{\partial t}\vartheta = c^2\Delta\vartheta + p \quad \Leftrightarrow \quad \frac{\partial}{\partial t}\vartheta - c^2\Delta\vartheta = p$$

Boundary and Initial conditions

To finally dertemine the temperature ϑ we must also know

- the initial temperature $\vartheta_0(\boldsymbol{x}) = \vartheta(\boldsymbol{x}, 0)$ for all $\boldsymbol{x} \in \Omega_0$
- whether or what kind of energy exchange occurs at the boundary $\partial\Omega_0$

Most common boundary conditions are (for the heat equation)

$$\boldsymbol{F} \cdot \boldsymbol{n} = \kappa(\boldsymbol{x}, t)(\vartheta - \vartheta_{\mathrm{a}})$$

with some ambient temperature ϑ_{a}
and the heat transfer coefficient κ ($[J/(m^2 sK)]$)

$\Rightarrow$ Fourier's law:

$$-\lambda \nabla\vartheta \cdot \boldsymbol{n} = -\lambda \partial_{\boldsymbol{n}}\vartheta = \kappa(\vartheta - \vartheta_{\mathrm{a}})$$

Boundary Conditions

These conditions are called **Robin boundary conditions**

$$\lambda \partial_{\boldsymbol{n}} \vartheta + \kappa(\vartheta - \vartheta_{\mathrm{a}}) = 0 \quad \text{on } \partial\Omega_0$$

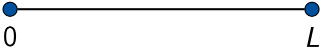
(Homogeneous) **Neumann condition**
$\kappa = 0$ is the limit case of perfect isolation and we obtain

$$\lambda \partial_{\boldsymbol{n}} \vartheta = 0 \quad \text{on } \partial\Omega_0$$

(Inhomogeneous) **Dirichlet condition**: $\vartheta = \vartheta_{\mathrm{a}}$
is the limit case $\kappa \to \infty$ is infinitely fast heat exchange

1. Solving the heat equation on a rod

Goal. Compute the temperature $u(t,x)$ of a (1D) rod or wire (infinitely thin, no heat source)



$$\begin{cases} \dfrac{\partial}{\partial t} u - c^2 \dfrac{\partial^2}{\partial x^2} u = 0 & \\ u(0,t) = u(L,t) = 0 & \text{Dirichlet boundary conditions} \\ u(x,0) = f(x) & \text{initial conditions (at time } 0\text{)} \end{cases}$$

Ansatz: Separation of variables

Ansatz. (or Idea: What if our) solution can be written as

$$u(x,t) = F(x)G(t)$$

We obtain

$$\frac{G'(t)}{c^2 G(t)} = -k = \frac{F''(x)}{F(x)} \qquad \text{for some constant } k \in \mathbb{R}$$

(we choose $-k$ just such that the following derivations are nicer)

or in other words two ODEs

$$F''(x) + kF(x) = 0$$
$$G'(t) + c^2 kG(t) = 0$$

Let's consider F first and distinguish different cases of k.

Case $k = 0$ in $F''(x) + kF(x) = 0$

Short summary of handwritten notes. Since $F''(x) = 0$ we have $F(x) = A + Bx$.
The boundary conditions $u(0, t) = u(L, t) = 0$ yield $F(x) = 0$, so

$$u(x, t) = 0 \quad \text{for all } x, t,$$

which is not an interesting solution.

Case $k < 0$ in $F''(x) + kF(x) = 0$

Short summary of handwritten notes. We have to solve a linear system starting from the linear combination of the fundamental solutions, but we also obtain $A = B = 0$ or $F(x) = 0$, so

$$u(x, t) = 0 \quad \text{for all } x, t,$$

which is (again) not an interesting solution.

Case $k > 0$ in $F''(x) + kF(x) = 0$

Short summary of handwritten notes. We first obtain that for for $k = \left(\frac{n\pi}{L}\right)^2$, $0 < n \in \mathbb{N}$ we can choose $F(x) = B\sin(\sqrt{k}x)$
Plugging these into $G'(t) + c^2G(t) = 0$ we obtain the general solutions

$$u_n(x,t) = F(x)G_n(t) = B_n \mathrm{e}^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

Towards a solution to the heat equation on the rod

Every superposition (linear combination) of the $u_n(x,t)$ is also a solution, so the general form is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$$

Question. How to determine the B_n?
Use the initial conditions $u(x,0) = f(x)$! We get

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

Do you recognise this? A Fourier series! To be precise of the odd extension f_o of $f(x), x \in [0,L]$!

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi}{L}x\right).$$

The solution to the heat equation on the rod

Theorem. Let $L > 0$ be given. The (1D) heat equation

$$\begin{cases} \dfrac{\partial}{\partial t} u - c^2 \dfrac{\partial^2}{\partial x^2} u = 0 & \\ u(0,t) = u(L,t) = 0 & \text{Dirichlet boundary conditions} \\ u(x,0) = f(x) & \text{initial conditions (at time } 0\text{)} \end{cases}$$

is solved by

$$u(x,t) = \sum_{n=1}^{\infty} B_n \mathrm{e}^{-\left(\frac{cn\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

with

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L} x\right).$$

2. Solution on an infinite rod

Let's consider

$$\begin{cases} \frac{\partial}{\partial t}u - c^2\frac{\partial^2}{\partial x^2}u = 0 & \\ \lim_{x\to\pm\infty} u(x,t) = 0 & \text{Dirichlet boundary conditions} \\ u(x,0) = f(x) & \text{initial conditions (at time 0) for} x \in \mathbb{R} \end{cases}$$

Idea. Since Fourier series worked for the bonded interval $[0, L]$, use Fourier transform here.

Fourier transform in x on both sides yields

$$\frac{\partial}{\partial t}\hat{u}(\omega,t) = -c^2\omega^2\hat{u}(\omega,t)$$

Since for any fixed ω this is an ODE in t we get

$$\hat{u}(\omega,t) = A(\omega)\mathrm{e}^{-c^2 2\omega^2 t}$$

Initial conditions and Inverse Fourier Transform

Using

$$\hat{u}(\omega, t) = A(\omega)\mathrm{e}^{-c^2 2\omega^2 t}$$

and the Fourier transform the initial conditions $f(x) = u(x, 0)$ we obtain

$$\hat{f}(\omega) = \hat{u}(\omega, 0) = A(\omega).$$

Thus we can compute $u(x, t)$ with the inverse Fourier transform

$$u(x, t) = \mathcal{F}^{-1}(\hat{u}(\omega, t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{-c^2\omega^2 t}\mathrm{e}^{i\omega x}\,\mathrm{d}\omega$$

Observation. for a certain $\hat{g}(\omega)$: **multiplication** in Fourier domain!

We obtain

$$u(x, t) = \mathcal{F}^{-1}(\hat{f}\hat{g}) = \frac{1}{\sqrt{2\pi}} f * \mathcal{F}^{-1}\left(\mathrm{e}^{-c^2\omega^2 t}\right)$$

Heat Kernel.

So we are left to compute $\mathcal{F}^{-1}(\mathrm{e}^{-c^2\omega^2 t})$.

From previously we know $\mathcal{F}(\mathrm{e}^{ax^2}) = \frac{1}{\sqrt{2a}}\mathrm{e}^{-\omega^2}2a$ we obtain with $a = \frac{1}{4c^2t}$ that

$$g(x) = \mathcal{F}^{-1}(\mathrm{e}^{-c^2\omega^2 t}) = \frac{\sqrt{2}}{c\sqrt{4t}}\mathrm{e}^{-\frac{x^2}{4c^2t}}$$

and hence

$$u(x,t) = f * \left(\frac{\sqrt{2}}{c\sqrt{4t}}\mathrm{e}^{-\frac{x^2}{4c^2t}}\right) = \int_{-\infty}^{\infty} f(v)\frac{1}{c\sqrt{4\pi t}}\mathrm{e}^{\frac{(x-v)^2}{4c^2t}}\,\mathrm{d}v$$

Definition. We define the Heat kernel by

$$G_\tau(x) = G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\mathrm{e}^{-\frac{x^2}{4\tau}}$$

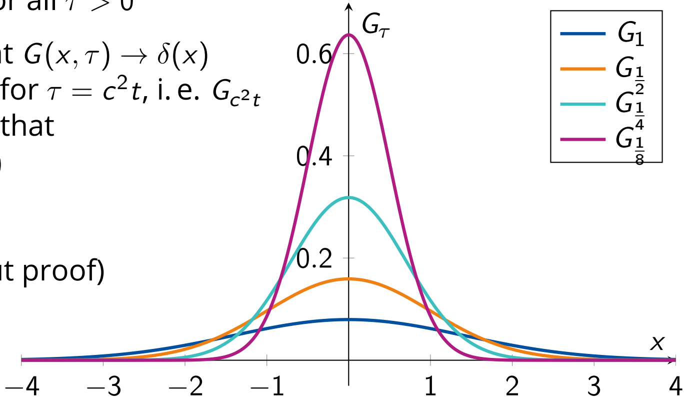
then we can write in short $u(x,t) = (f * G_{c^2t})(x)$.

Observations on the heat kernel

For the heat kernel $G_\tau = G(x,\tau) = \frac{1}{\sqrt{4\pi\tau}}\mathrm{e}^{-\frac{x^2}{4\tau}}$ we have

- $\displaystyle\int_{-\infty}^{\infty} G(x,\tau)\,\mathrm{d}x = 1$ for all $\tau > 0$
- For $\tau \to 0$ we get that $G(x,\tau) \to \delta(x)$
- The same of course for $\tau = c^2 t$, i. e. $G_{c^2 t}$

$\Rightarrow$ For $t \to 0$ we obtain that
$u(x,t) = (f * G_{c^2 t})(x)$
converges to
$(f * \delta)(x) = f(x)$.
(though here without proof)



Solution for the infinite rod.

Theorem. The heat equation on the x-axis

$$\begin{cases} \frac{\partial}{\partial t} u - c^2 \frac{\partial^2}{\partial x^2} u = 0 & \\ \lim_{x\to\pm\infty} u(x,t) = 0 & \text{Dirichlet boundary conditions} \\ u(x,0) = f(x) & \text{initial conditions (at time 0) for} x \in \mathbb{R} \end{cases}$$

can be solved by

$$u(x,t) = (f * G_{c^2 t})(x) = \int_{-\infty}^{\infty} f(v) \frac{1}{c\sqrt{4\pi t}} \mathrm{e}^{\frac{(x-v)^2}{4c^2 t}} \, \mathrm{d}v$$

and we have that $\lim\limits_{t\to 0} u(x,t) = f(x)$

3. Laplace equation

Goal. Find an equilibrium state.

1. temperature will remain steady/not change over time
   $\Rightarrow \dfrac{\partial}{\partial t} u(\boldsymbol{x}, t) = 0$
   and no need for initial conditions (in time)
2. Thus the temperature field u at equilibrium state satisfies the Laplace equation
   $$-c^2 \Delta u = 0$$

If the right hand side is not 0 we obtain the Poisson problem.

For the case of a 2D problem (and $c = 1$) we obtain

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

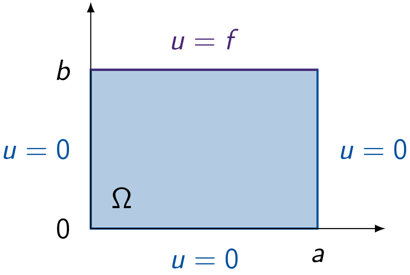
($+$ boundary conditions)

4.1 The Laplace equation on a rectangular domain

We consider the Laplace equation

$$\Delta u = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

on a bounded rectangular domain



- $\Omega = [0, a] \times [0, b]$
- $u(x, 0) = u(0, y) = u(a, y) = 0$
- $u(x, b) = f(x)$

Ansatz. Use (again) separation of variables:
$u(\boldsymbol{x}, t) = u(\boldsymbol{x}) = F(x)G(y)$.

Solution of the Laplace equation on a rectangle

The solution to the Laplace equation on a bounded ractangle reads

$$u(x, y) = \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi}{a} y\right) \sin\left(\frac{n\pi}{a} x\right)$$

where from the boundary condition $u(x, b) = f(x)$ we obtain that

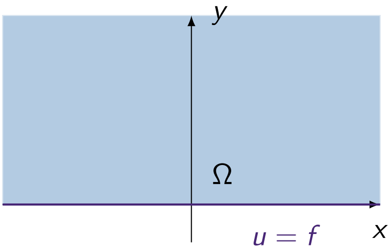
$$A_n = \left(\sinh \frac{n\pi b}{a}\right)^{-1} \frac{2}{a} \int_0^a f(x) \sin\left(\frac{n\pi}{a} x\right) \mathrm{d}x.$$

The Laplace equation in the half-plane

We consider the Laplace equation

$$\frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u = 0$$

on the half-plane



- $\Omega = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$
- $\lim_{x \to \pm\infty} u(x, y) = 0$ for any $y \geq 0$
- $\lim_{y \to \infty} u(x, y) = 0$ for any x
- $u(x, 0) = f(x)$

Idea. Use Fourier transform (w.r.t. x), we write $\mathcal{F}_x$ on our PDE

Towards the solution of the Laplace equation

$$\hat{u}(\omega, y) = \hat{f}(\omega)\mathrm{e}^{-|\omega|y}$$

Using the inverse Fourier transform, we obtain

$$u(x, y) = \mathcal{F}_x^{-1}\left(\hat{f}(\omega)\mathrm{e}^{-|\omega|y}\right)$$

Introducing a function $g(x) := \mathcal{F}_x^{-1}(\mathrm{e}^{-|\omega|y})$ we can use the convolution theorem to see that

$$u(x, y) = \frac{1}{\sqrt{2\pi}} f * g$$

and since $\mathcal{F}_x^{-1}(e^{-|\omega|y}) = \sqrt{\frac{2}{\pi}}\frac{y}{y^2+x^2}$ we obtain

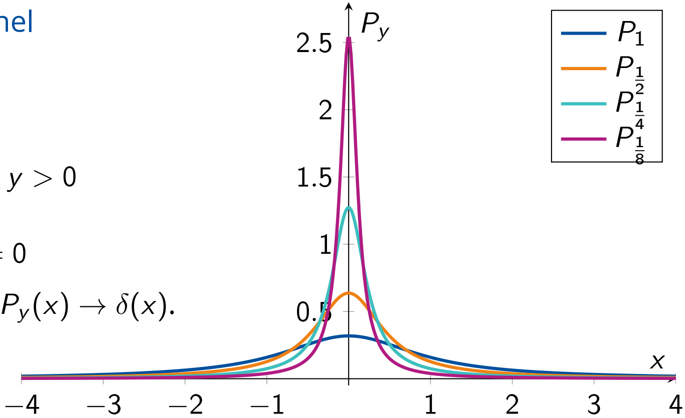
$$u(x, y) = \frac{1}{\sqrt{2\pi}} f * \left(\sqrt{\frac{2}{\pi}}\frac{y}{y^2+x^2}\right) = \int_{-\infty}^{\infty} f(t) \cdot \frac{1}{\pi} \cdot \frac{y}{y^2 + (x-t)^2}\,\mathrm{d}t$$

The Poisson kernel

Definition. The function

$$P_y(x) = \frac{1}{\pi} \frac{y}{y^2 + x^2}$$

is called the Poisson kernel for the upper half space.



Observations.

- $\int_{-\infty}^{\infty} P_y(x) = 1$ for all $y > 0$
- $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) P_y(x) = 0$
- for $y \to 0$ we obtain $P_y(x) \to \delta(x)$.

$\Rightarrow$ we expect $\lim_{y\to 0} u(x, y) = \lim_{y\to 0} (f * P_y)(x) = (f * \delta)(x) = f(x)$.

Summary

heat equation

one-dimensional & time $u(x,t)$

$$\frac{\partial}{\partial t}u - c^2\frac{\partial^2}{\partial x^2}u = 0$$

- boundary conditions (in x)
- initial conditions (in $t=0$): $f(x)$

$$\Rightarrow u(x,t) = (f * G_{c^2t})(x)$$

- $\int_{-\infty}^{\infty} G_{c^2t}(x)\,\mathrm{d}x = 1$
- $\lim_{t\to 0} G_{c^2t}(x) = \delta(x)$
- $G_{c^t}(x)$ fulfils the heat equation

Laplace equation

2D equilibrium $u(x,y)$

$$-\frac{\partial^2}{\partial x^2}u - \frac{\partial^2}{\partial y^2}u = 0$$

- boundary conditions (in x)
- boundary conditions in y: $f(x)$

$$\Rightarrow u(x,y) = (f * P_y)(x)$$

- $\int_{-\infty}^{\infty} P_y(x)\,\mathrm{d}x = 1$
- $\lim_{y\to 0} P_y(x) = \delta(x)$
- $P_y(x)$ fulfils the Laplace equation

$\Rightarrow$ for both we convolve f a solution with the given PDE