

TMA4125 Matematikk 4N

Wave Equation

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The Wave Equation

the multi-one-dimensional wave equation is the PDE for some L, c > 0

$$\begin{cases} \frac{\partial^2}{\partial t^2} u = c^2 \Delta u & \mathbf{x} \in \Omega \subset \mathbb{R}^d \\ u(\mathbf{x}, t) = 0 & \mathbf{x} \in \partial \Omega \quad \text{(boundary conditions)} \\ u(\mathbf{x}, 0) = f(\mathbf{x}) & \text{(initial condition)} \\ \frac{\partial}{\partial t} u(\mathbf{x}, 0) = g(\mathbf{x}) & \text{(initial condition)} \end{cases}$$

Note. For the wave equation we need two initial conditions:

- The initial position $f(\mathbf{x}x)$
- The initial velocity $g(\mathbf{x}x)$

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The Wave Equation

the multi-one-dimensional wave equation is the PDE for some L, c > 0

$$\begin{cases} \frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \mathbf{x} \in \partial \Omega \quad \text{(boundary conditions)} \\ u(x, 0) = f(x) & \text{(initial condition)} \\ \frac{\partial}{\partial t} u(x, 0) = g(x) & \text{(initial condition)} \end{cases}$$

Note. For the wave equation we need two initial conditions:

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Interactive 5 Min I: What are the steps to solve the PDE?

Let's collect – for maybe about 5 minutes – the general approach to solve the PDE. Answer in "1. …" "2. …" and so on

We collect the ideas in Mentimeter

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Interactive 5 Min II: Which to ODEs to we obtain?

•
$$F'' + kF = 0$$
 and $G' - \frac{k}{c^2}G = 0$

•
$$F' + kF = 0$$
 and $G'' - \tilde{c}^2 kG = 0$

•
$$F'' + kF = 0$$
 and $G'' + c^2 kG = 0$

•
$$F'' + kF = 0$$
 and $G'' - ckG = 0$

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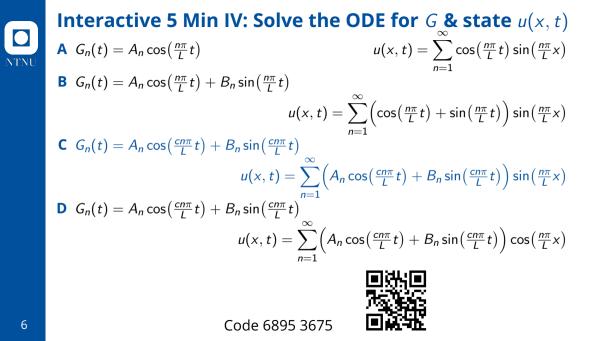


Interactive 5 Min III: Solve the ODE we got for *F*.

Hint. Take a look at last week and the heat equation.

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What's different in the initial conditions?

Let's collect ideas - What's left to do?

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right) \right) \sin\left(\frac{n\pi}{L}x\right)$$



Summary. Solution to the wave equation

The one-dimensional wave equation from the first slide has the solution

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi}{L}t\right) + B_n \sin\left(\frac{cn\pi}{L}t\right) \right) \sin\left(\frac{n\pi}{L}x\right)$$

with

$$A_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$$
$$B_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$



Notation & Terminology

The single functions

$$u_n(x,t)\Big(A_n\cos\left(\frac{cn\pi}{L}t\right)+B_n\sin\left(\frac{cn\pi}{L}t\right)\Big)\sin\left(\frac{n\pi}{L}x\right)$$

are called eigenfunctions of the wave equation.

The values

$$\lambda_n = \frac{cn\pi}{L}, n = 1, 2, \dots$$

are called eigenvalues and the set $\{\lambda_1, \lambda_2, \ldots\}$ is called the spectrum.

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Alternative approach. D'Alembert's formula

The one-dimensional wave wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} u = c^2 \frac{\partial^2}{\partial x^2} u & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \text{(boundary conditions)} \\ u(x, 0) = f(x) & \text{(initial condition)} \\ \frac{\partial}{\partial t} u(x, 0) = g(x) & \text{(initial condition)} \end{cases}$$

Can also be solved with the following Ansatz:

Let φ,ψ be two functions that are twice differentiable functions. Then we consider the function

$$u(x,t) = \varphi(x+ct) + \psi(x-ct).$$

It actually fulfils the first equation above.