TMA4125 Matematikk 4N

Wave Equation

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The Wave Equation

the multi-one-dimensional wave equation is the PDE for some $L, c > 0$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u = c^2 \Delta u & \boldsymbol{x} \in \Omega \subset \mathbb{R}^d \\ u(\boldsymbol{x}, t) = 0 & \boldsymbol{x} \in \partial\Omega \quad \text{(boundary conditions)} \\ u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) & \text{(initial condition)} \\ \dfrac{\partial}{\partial t} u(\boldsymbol{x}, 0) = g(\boldsymbol{x}) & \text{(initial condition)} \end{cases}$$

Note. For the wave equation we need two initial conditions:

- The initial position $f(\boldsymbol{x}x)$
- The initial velocity $g(\boldsymbol{x}x)$

The Wave Equation

the multi-one-dimensional wave equation is the PDE for some $L, c > 0$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u = c^2 \dfrac{\partial^2}{\partial x^2} u & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \boldsymbol{x} \in \partial\Omega \quad \text{(boundary conditions)} \\ u(x, 0) = f(x) & \text{(initial condition)} \\ \dfrac{\partial}{\partial t} u(x, 0) = g(x) & \text{(initial condition)} \end{cases}$$

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Interactive 5 Min I: What are the steps to solve the PDE?

Let's collect – for maybe about 5 minutes – the general approach to solve the PDE.

Answer in "1. …" "2. …" and so on

We collect the ideas in Mentimeter

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Interactive 5 Min II: Which to ODEs to we obtain?

- $F'' + kF = 0$ and $G' - \frac{k}{c^2} G = 0$
- $F' + kF = 0$ and $G'' - c^2 kG = 0$
- $F'' + kF = 0$ and $G'' + c^2 kG = 0$
- $F'' + kF = 0$ and $G'' - ckG = 0$

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Interactive 5 Min III: Solve the ODE we got for F.

Hint. Take a look at last week and the heat equation.

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Interactive 5 Min IV: Solve the ODE for G & state $u(x,t)$

A $G_n(t) = A_n \cos\left(\frac{n\pi}{L} t\right)$

$$u(x,t) = \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L} t\right) \sin\left(\frac{n\pi}{L} x\right)$$

B $G_n(t) = A_n \cos\left(\frac{n\pi}{L} t\right) + B_n \sin\left(\frac{n\pi}{L} t\right)$

$$u(x,t) = \sum_{n=1}^{\infty} \left(\cos\left(\frac{n\pi}{L} t\right) + \sin\left(\frac{n\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

C $G_n(t) = A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)\right) \sin\left(\frac{n\pi}{L} x\right)$$

D $G_n(t) = A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)$

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right)\right) \cos\left(\frac{n\pi}{L} x\right)$$

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What's different in the initial conditions?

Let's collect ideas – What's left to do?

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos\left(\tfrac{cn\pi}{L} t\right) + B_n \sin\left(\tfrac{cn\pi}{L} t\right) \right) \sin\left(\tfrac{n\pi}{L} x\right)$$

Summary. Solution to the wave equation

The one-dimensional wave equation from the first slide has the solution

$$u(x,t)=\sum_{n=1}^{\infty}\Big(A_n\cos\big(\tfrac{cn\pi}{L}t\big)+B_n\sin\big(\frac{cn\pi}{L}t\big)\Big)\sin\big(\tfrac{n\pi}{L}x\big)$$

with

$$A_n=\frac{2}{L}\int_0^L f(x)\sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x$$
$$B_n=\frac{2}{cn\pi}\int_0^L g(x)\sin\Big(\frac{n\pi}{L}x\Big)\,\mathrm{d}x$$

Notation & Terminology

The single functions

$$u_n(x,t)\left(A_n\cos\left(\tfrac{cn\pi}{L}t\right)+B_n\sin\left(\tfrac{cn\pi}{L}t\right)\right)\sin\left(\tfrac{n\pi}{L}x\right)$$

are called eigenfunctions of the wave equation.

The values

$$\lambda_n=\frac{cn\pi}{L},\, n=1,2,\ldots$$

are called eigenvalues and the set $\{\lambda_1,\lambda_2,\ldots\}$ is called the spectrum.

Alternative approach. D'Alembert's formula

The one-dimensional wave wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u = c^2 \dfrac{\partial^2}{\partial x^2} u & x \in [0, L] \\ u(0, t) = u(L, t) = 0 & \text{(boundary conditions)} \\ u(x, 0) = f(x) & \text{(initial condition)} \\ \dfrac{\partial}{\partial t} u(x, 0) = g(x) & \text{(initial condition)} \end{cases}$$

Can also be solved with the following Ansatz:
Let φ, ψ be two functions that are twice differentiable functions. Then we consider the function

$$u(x, t) = \varphi(x + ct) + \psi(x - ct).$$

It actually fulfils the first equation above.