TMA4125 Matematikk 4N

A (very) short Summary

Ronny Bergmann

Department of Mathematical Sciences, NTNU.

April 27, 2022

Norwegian University of Science and Technology

1. Introduction and Preliminaries

Let's do a short recapitulation of the topics we covered in TMA4125 In the first week we discussed a few concepts from earlier classes

- real and complex vector spaces, e. g. of polynomials
- norm, scalar product and orthogonality
- Cauchy-Schwartz inequality and Gram-Schmidt orthogonalisation
- orthogonal projection and best-approximation
- Taylor expansion, $\mathcal{O}$ notation

These will not directly be asked in the exam, but we use them often

The following slides state the key concepts/topics we covered.

2. Polynomial Interpolation

Goal. Find a function f that describes this data, i. e.

$$f(x_i) = y_i \qquad i = 0, \ldots, n,$$

- Statement of the Interpolation problem
- Lagrange interpolation based on Lagrange Polynomials (Cardinal functions)
- Newton Interpolation using Newton polynomials & Newton divided differences
- Estimate the interpolation Error.
- Runge's example and optimal distribution of interpolation nodes (Chebyshev nodes/interpolation)

3. Numerical Integration

Goal. Solve the task of integration by using a numerical quadrature

$$I[f](a,b) := \int_a^b f(x)\,\mathrm{d}x \approx Q[f](a,b) = \sum_{i=0}^{n} w_i f(x_i),$$

using certain points $x_i \in [a,b]$ and weights w_i.

- Midpoint rule, trapezoidal rule, Simpson rule, Gauß-Legendre
- Quadrature from integrating interpolation polynomials
- Degree of exactness, error estimates

NTNU

3. Numerical Integration

Goal. Solve the task of integration by using a numerical quadrature

$$I[f](a,b) := \int_a^b f(x)\,\mathrm{d}x \approx Q[f](a,b) = \sum_{i=0}^{n} w_i f(x_i),$$

using certain points $x_i \in [a,b]$ and weights w_i.

- Midpoint rule, trapezoidal rule, Simpson rule, Gauß-Legendre
- Quadrature from integrating interpolation polynomials
- Degree of exactness, error estimates
- Composite Quadrature & Error estimates
- (shortly) Experimental order of convergence (EOC)
- Adaptive Integration
- Newton–Côtes Formulae, Gauß–Quadrature (not relevant for the exam)

4. Nonlinear Equations

Goal. Numerically find the solution to a nonlinear equation

$$f(x) = 0 \qquad \text{or for the multivariate case } \boldsymbol{f}(\boldsymbol{x}) = 0.$$

- Existence and uniqueness of solutions
- Bisection Method
- Fix point iteration: Existence, Uniqueness, Convergence
- Newton's Method: Error Analysis, Convergence

NTNU

5. Laplace Transform

$$F(s) = \mathcal{L}(f)(s) \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t$$

- s-shifting & t-shifting
- existence & uniqueness of the Laplace transform
- Laplace transform of derivatives and integrals
- solving ODEs with the Laplace transform
- unit step function, Dirac's delta "function"
- Convolution & Integral Equations
- Systems of ODEs

6. Numerical Methods for Solving ODEs

- Initial value problem for ordinary differential equations (ODE)
- Euler’s method and Heun’s method, derivation of the methods and implementation
- Taylor method, General one-step methods
- Error theory including consistency/convergence error, convergence order
- Explicit Runge-Kutta methods: Motivation, description via Butcher tables, implementation Adaptive/embedded Runge-Kutta methods including computable error estimates and adaptive time-step selection
- Implicit Methods: Implicit Euler
- Stability & Stability region.

NTNU

7. Fourier Series

Goal. Describe a (2π-)periodic function f as

$$f \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx} = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

- composing and decomposing functions
- trigonometric polynomials, vector spaces
- Orthogonal projection and best approximation (again)
- Convergence of the Fourier series, Gibbs phenomenon
- Real and complex Fourier Series, Sine and Cosine series
- $2L$-periodic functions, even & odd extension
- Parseval identity, Spektrum and Amplitude
- Convolution, Derivatives & further properties

8. (Continuous) Fourier Transform

Goal. Extend Fourier Series idea to the whole real line

$$\hat{f}(\omega) = \mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\omega x}\,\mathrm{d}x$$

similarly: Fourier Sine and Fourier Cosine transform

- derivation of the (Continuous) Fourier Transform
- Linearity, Derivatives, Convolution
- Inverse Fourier Transform

8. Discrete Fourier Transform

Goal. Compute Fourier Series on the computer: Discrete Fourier Transform (DFT), i. e. for a signal $\boldsymbol{f} = (f_0, \ldots, f_{N-1})^{\mathrm{T}} \in \mathbb{C}^N$ compute

$$\hat{f}_k = \sum_{j=0}^{N-1} f_j \, \mathrm{e}^{-2\pi \mathrm{i} jk/N}, \qquad k = 0, \ldots, N-1.$$

or shorter in matrix-vector form

$$\hat{\boldsymbol{f}} = \mathcal{F}_N \boldsymbol{f} \quad \text{with } \mathcal{F}_N = \left(\mathrm{e}^{-2\pi \mathrm{i} jk/N}\right)_{j,k=0}^{N-1} = \left(w_N^{jk}\right)_{j,k=0}^{N-1}$$

- Aliasing
- FFT-shift and interpretation of discrete Fourier coefficients
- Application: Denoising / Filtering out a frequency.
- Discrete Sine and Cosine Transforms, FFT (short)

NTNU

9. Heat Equation

$$\begin{cases} \frac{\partial}{\partial t} u - c^2 \frac{\partial^2}{\partial x^2} u = 0 & \\ u(0,t) = u(L,t) = 0 & \text{Dirichlet boundary conditions} \\ u(x,0) = f(x) & \text{initial conditions (at time 0)} \end{cases}$$

- Ansatz: separation of variables
- Fundamental theorem on superposition
- Solution of the heat equation by Fourier series
- Steady two-dimensional heat equation: Laplace's equation
- Heat equation: Modelling very long bars
- Solving the Dirichlet problem in a rectangle

10. Wave Equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u = c^2 \Delta u & \boldsymbol{x} \in \Omega \subset \mathbb{R}^d \\ u(\boldsymbol{x}, t) = 0 u(\boldsymbol{x}, 0) = f(\boldsymbol{x}) & \text{(initial condition)} \\ \dfrac{\partial}{\partial t} u(\boldsymbol{x}, 0) = g(\boldsymbol{x}) & \text{(initial condition)} \end{cases}$$

- Formulation of wave equation with boundary and initial conditions
- Solution formula for a wave equation on a bounded interval using separation of variables
- d'Alembert's solution formula for wave equation on real line

11. Numerical Methods for Solving PDEs

- Formulation of general two-point value problems
- Finite difference operators for first and second order derivatives and their approximation/convergence order
- Finite difference methods for general two-point value problem with Dirichlet boundary conditions
- Finite difference methods for general two-point value problem with Neuman/Robin boundary conditions
- Finite difference methods for the heat equation, Method of Lines and Stability

NTNU