SIF5013 Matematikk 4N
15. mai 2000

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Løsningsforslag

Oppgavesettet har 12 punkter, 1abc, 2abc, 3ab, 4, 5, 6, 7, som teller likt ved bedømmelsen.

1 **a)**

$$g(t) = (t-\pi)u(t-\pi) = f_1(t-\pi)u(t-\pi), \quad f_1(t) = t$$

$$G(s) = \mathcal{L}(g) = F_1(s)e^{-\pi s} = \frac{1}{s^2}e^{-\pi s} \quad \text{ved skiftteorem 2, eller, alternativt,}$$

$$G(s) = \int_0^\infty g(t)e^{-st}\,dt = \int_\pi^\infty (t-\pi)e^{-st}\,dt = \Big[-\frac{t-\pi}{s}e^{-st} - \frac{1}{s^2}e^{-st}\Big]_\pi^\infty = \frac{1}{s^2}e^{-\pi s}$$

$$H(s) = \frac{s+1}{(s+1)^2+1} = F_2(s+1), \quad F_2(s) = \frac{s}{s^2+1}$$

$$h(t) = \mathcal{L}^{-1}(H) = e^{-t}f_2(t) = e^{-t}\cos t \quad \text{ved skiftteorem 1}$$

b)

$$Y(s) = \mathcal{L}\{y(t)\}, \quad y(t) + t * y(t) = g(t) \Rightarrow Y + \frac{1}{s^2}\cdot Y = \frac{1}{s^2}e^{-\pi s}$$

$$Y(s) = \frac{s^2}{s^2+1}\cdot\frac{1}{s^2}e^{-\pi s} = \frac{1}{s^2+1}e^{-\pi s}$$

$$y(t) = \mathcal{L}^{-1}(Y) = \sin(t-\pi)u(t-\pi) = -u(t-\pi)\sin t$$

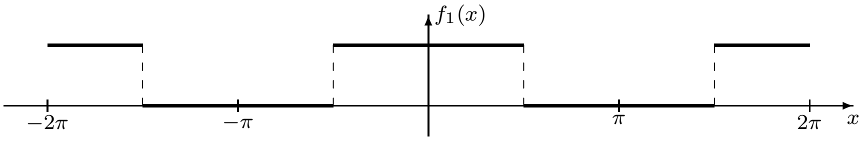
c)

$$Y(s) = \mathcal{L}\{y(t)\}, \qquad s^2Y - s - (-1) + 2(sY-1) + 2Y = e^{-\pi s}$$

$$Y(s) = \frac{s+1}{s^2+2s+2} + \frac{1}{s^2+2s+2}e^{-\pi s} = H(s) + \frac{1}{(s+1)^2+1}e^{-\pi s}$$

$$y(t) = \mathcal{L}^{-1}(Y) = h(t) + \big[e^{-(t-\pi)}\sin(t-\pi)\big]u(t-\pi) = e^{-t}\cos t - e^{\pi-t}u(t-\pi)\sin t$$

2 **a)**



$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\int_0^{\pi/2} dx = \frac{1}{2}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi/2}\cos nx\,dx = \frac{2}{\pi}\frac{\sin nx}{n\pi}\Big|_0^{\pi/2} = \frac{2\sin(n\pi/2)}{n\pi}$$

$$a_n = 0 \text{ for } n \text{ partall}, \quad a_n = \frac{2}{n\pi} \text{ for } n = 1, 5, \dots, \quad a_n = \frac{-2}{n\pi} \text{ for } n = 3, 7, \dots$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\Big(\cos x - \frac{\cos 3x}{3} + \frac{\cos 5x}{5} - + \cdots\Big) = \frac{1}{2} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m \cos(2m+1)x}{2m+1}$$

b)

$$u(x,t) = F(x)\cdot G(t) \quad \text{innsettes i (1)}:$$

$$F''G = FG + FG', \quad \frac{F''}{F} = 1 + \frac{G'}{G} = k \quad \text{(konstant)}$$

$$\text{(I)} \quad F'' - kF = 0, \quad F'(0) = 0,\ F'(\pi) = 0, \qquad \text{(II)} \quad G' - (k-1)G = 0$$

(I) $k > 0,\ k = \mu^2: \quad F(x) = Ae^{\mu x} + Be^{-\mu x},\ F'(x) = \mu Ae^{\mu x} - \mu Be^{-\mu x}$

$F'(0) = F'(\pi) = 0 \Rightarrow A = B = 0,\ F(x) = 0$

$k = 0: \quad F(x) = A + Bx,\ F'(x) = B$

$F'(0) = F'(\pi) = 0 \Rightarrow B = 0,\ F(x) = 1,\ (A = 1)$

$k < 0,\ k = -p^2: \ F(x) = A\cos px + B\sin px,\ F'(x) = -pA\sin px + pB\cos px$

$F'(0) = 0 \Rightarrow B = 0,\ F'(\pi) = 0,\ A \neq 0 \Rightarrow p = n, \quad F(x) = \cos nx,\ (A = 1)$

$F(x) = \cos nx,\ n = 0, 1, 2, 3, \ldots$

(II) $k = -n^2,\ n = 0, 1, 2, \ldots: \quad G' + (n^2+1)G = 0,\ G(t) = Ce^{-(n^2+1)t}$

Løsningene av (1) på formen $u(x,t) = F(x)G(t)$ som tilfredsstiller (2):

$$u_n(x,t) = C_n e^{-(n^2+1)t}\cos nx, \quad C_n \text{ vilkårlig konstant, } n = 0,\ 1,\ 2,\ \ldots$$

c) Superposisjonprinsippet:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = \sum_{n=0}^{\infty} C_n e^{-(n^2+1)t}\cos nx \quad \text{oppfyller (1) og (2)}$$

$$f(x) = u(x,0) = \sum_{n=0}^{\infty} C_n \cos nx \ \Rightarrow\ C_n = a_n \overset{\mathbf{a)}}{=} \begin{cases} 1/2 & \text{for } n = 0 \\ 2/(n\pi) & \text{for } n = 1, 5, \ldots \\ -2/(n\pi) & \text{for } n = 3, 7, \ldots \end{cases}$$

$$u(x,t) = \frac{1}{2}e^{-t} + \frac{2}{\pi}\Big(e^{-2t}\cos x - \frac{1}{3}e^{-(3^2+1)t}\cos 3x + \frac{1}{5}e^{-(5^2+1)t}\cos 5x - + \cdots\Big)$$

$$= \frac{1}{2}e^{-t} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{2m+1}e^{-[(2m+1)^2+1]t}\cos(2m+1)x, \quad 0 \leq x \leq \pi,\ t > 0$$

Omformer: $2\cos x \cdot \cos 2x = \cos(x - 2x) + \cos(x + 2x) = \cos x + \cos 3x$

$$\cos x + \cos 3x = u(x,0) = \sum_{n=0}^{\infty} C_n \cos nx \ \Rightarrow\ C_1 = 1,\ C_3 = 1,\ C_n = 0 \text{ ellers}$$

$$u(x,t) = e^{-(1^2+1)t}\cos x + e^{-(3^2+1)t}\cos 3x = e^{-2t}\cos x + e^{-10t}\cos 3x$$

$\boxed{3}$ **a)**

$x = 0:$

$$\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}\right] = f(0) = 1 \;\Rightarrow\; \sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{\pi}{2\sinh\pi}-\frac{1}{2}$$

$x = \pi:$

$$\frac{2\sinh\pi}{\pi}\left[\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2}(-1)^n\right] = \frac{f(\pi+0)+f(\pi-0)}{2} = \frac{e^{\pi}+e^{-\pi}}{2} = \cosh\pi$$

Følgelig: $\displaystyle \frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{1+n^2} = \frac{\pi}{2\sinh\pi}\cdot\cosh\pi$ dvs. $\displaystyle \sum_{n=1}^{\infty}\frac{1}{1+n^2} = \frac{\pi\cosh\pi}{2\sinh\pi}-\frac{1}{2}$

b)

Invers Fouriertransformert: $\displaystyle \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\cos(\pi w/2)}{1-w^2}e^{iwx}\,dw = f(x)$ for alle x

$x = 0:$ $\displaystyle \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\,dw = f(0) = 1$ dvs. $\displaystyle \int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\,dw = \pi$

$x = \pi/2:$ $\displaystyle \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}e^{i\pi w/2}\,dw = f(\pi/2) = 0$

Eulers formel gir $\displaystyle \int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\cos\frac{\pi w}{2}\,dw + i\underbrace{\int_{-\infty}^{\infty}\frac{\cos(\pi w/2)}{1-w^2}\sin\frac{\pi w}{2}\,dw}_{0} = 0$

og følgelig: $\displaystyle \int_{-\infty}^{\infty}\frac{\cos^2(\pi w/2)}{1-w^2}\,dw = 0$

$\boxed{4}$ Gitt er tre punkter, altså er polynomet av grad to,

$$p_2(x) = \sum_{j=0}^{2} f_j L_j(x) = L_1(x)$$

$$L_1(x) = \frac{(x-0)(x-\pi)}{(\pi/2-0)(\pi/2-\pi)} = -4\frac{x(x-\pi)}{\pi^2} \; .$$

Punktene er jevnt fordelt, altså er

$$|\sin(x)-p_2(x)| \le \frac{1}{4(2+1)}\left(\frac{\pi}{2}\right)^3 = 0.3230 \; .$$

$\boxed{5}$

$$T_{\pi/8} = \frac{\pi}{16}\left(\tan(0)+2\tan(\pi/8)+\tan(\pi/4)\right) = \frac{\pi}{16}\left(2\sqrt{2}-1\right) = 0.3590$$

$$\left|T_h - \int_0^{\pi/4}\tan(x)\,dx\right| \le \frac{M_2}{12}\frac{(b-a)^3}{n^2} = \frac{M_2}{12}(b-a)h^2 \;, \quad h = \frac{b-a}{n} \; .$$

Vi har

$$f(x) = \tan(x) \;, \quad f'(x) = \frac{1}{\cos^2(x)} \;, \quad f''(x) = 2\frac{\sin(x)}{\cos^3(x)} \; .$$

Telleren i $f"(x)$ vokser og nevneren avtar i $[0, \pi/4]$. Dermed er $f"(x)$ voksende, og

$$|f''(x)| \leq M_2 = 2\frac{\sin(\pi/4)}{\cos^3(\pi/4)} = 4 \ .$$

Betingelsen

$$\left| T_h - \int_0^{\pi/4} \tan(x)\, dx \right| \leq \frac{M_2}{12}\frac{\pi}{4}h^2 \leq 10^{-4}$$

oppfylles altså med

$$h \leq \sqrt{\frac{12}{\pi}} 10^{-2} = 1.9544 \cdot 10^{-2} < 0.02 \ .$$

$\boxed{6}$ Ved innføring av nye variable

$$Y_1 = x_1 \ , \quad Y_2 = x_2 \ , \quad Y_3 = x_1' \ , \quad Y_4 = x_2'$$

finner vi at

$$Y' = \begin{pmatrix} x_1' \\ x_2' \\ -\frac{k_1}{m_1}x_1 + \frac{k_2}{m_1}(x_2 - x_1) - \frac{d}{m_1}x_1' \\ -\frac{k_2}{m_2}(x_2 - x_1) - \frac{k_3}{m_2}x_2 - \frac{d}{m_2}x_2' \end{pmatrix} \ ,$$

altså

$$Y' = \begin{pmatrix} Y_3 \\ Y_4 \\ -\frac{k_1}{m_1}Y_1 + \frac{k_2}{m_1}(Y_2 - Y_1) - \frac{d}{m_1}Y_3 \\ -\frac{k_2}{m_2}(Y_2 - Y_1) - \frac{k_3}{m_2}Y_2 - \frac{d}{m_2}Y_4 \end{pmatrix} =: F(Y)$$

Startbetingelsene blir da

$$Y(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ x_1'(0) \\ x_2'(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \ .$$

$\boxed{7}$

$$K_1 = hF(Y(0)) = 0.1 \begin{pmatrix} 2 - 2 \cdot 2 \\ 2 \cdot 2 - 2 \end{pmatrix} = \begin{pmatrix} -0.2 \\ 0.2 \end{pmatrix} \ ,$$

$$K_2 = hF(Y(0) + K_1) = 0.1F \begin{pmatrix} 1.8 \\ 2.2 \end{pmatrix} = 0.1 \begin{pmatrix} 1.8 - 1.8 \cdot 2.2 \\ 1.8 \cdot 2.2 - 2.2 \end{pmatrix} = \begin{pmatrix} -0.216 \\ 0.176 \end{pmatrix} \ ,$$

$$Y(0.1) \approx Y(0) + \frac{1}{2}(K_1 + K_2) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} -0.416 \\ 0.376 \end{pmatrix} = \begin{pmatrix} 1.792 \\ 2.188 \end{pmatrix} \ .$$