



Norges teknisk-naturvitenskapelige universitet  
 Institutt for matematiske fag

SIF5016 Matematikk 4N  
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Løsningsforslag

Opgavesettet har 11 punkter: 1, 2ab, 3ab, 4ab, 5, 6, 7ab som teller likt ved bedømmelsen.

1 a) Finner ligningen for  $f(t)$  fra grafen, og bruker formelene for Fourierkoeffisientene:

$$f(t) = \begin{cases} -\pi - t & \text{for } -\pi < t < 0 \\ t & \text{for } 0 < t < \pi \end{cases}$$

$$a_0 = 0 \quad (\text{av grafen ser vi at } \int_{-\pi}^{\pi} f(t) dt = 0)$$

$$\begin{aligned} a_n &= \frac{1}{\pi} \left\{ \int_{-\pi}^0 -(t+\pi) \cos nt dt + \int_0^{\pi} t \cos nt dt \right\} \\ &= \frac{1}{\pi} \left\{ \underbrace{\left[ -(t+\pi) \frac{\sin nt}{n} \right]_{-\pi}^0}_{0} + \int_{-\pi}^0 \frac{\sin nt}{n} dt + \left[ t \frac{\sin nt}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin nt}{n} dt \right\} \\ &= \frac{1}{\pi} \left\{ \left[ -\frac{\cos nt}{n^2} \right]_{-\pi}^0 + \left[ \frac{\cos nt}{n^2} \right]_0^{\pi} \right\} = \frac{2(-1)^n - 1}{\pi n^2} = \begin{cases} 0 & \text{for } n \text{ partall} \\ -4/(\pi n^2) & \text{for } n \text{ oddetall} \end{cases} \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{\pi} \left\{ \int_{-\pi}^0 -(t+\pi) \sin nt dt + \int_0^{\pi} t \sin nt dt \right\} \\ &= \frac{1}{\pi} \left\{ \left[ (t+\pi) \frac{\cos nt}{n} \right]_{-\pi}^0 - \int_{-\pi}^0 \frac{\cos nt}{n} dt - \left[ t \frac{\cos nt}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{\cos nt}{n} dt \right\} \\ &= \frac{1}{\pi} \left\{ \pi \frac{1}{n} - \pi \frac{(-1)^n}{n} \right\} = \frac{1 - (-1)^n}{n} = \begin{cases} 0 & \text{når } n \text{ er partall} \\ 2/n & \text{når } n \text{ er oddetall} \end{cases} \end{aligned}$$

$$f(t) \sim \sum_{\substack{n=1 \\ n \text{ odde}}}^{\infty} \left( \frac{2}{n} \sin nt - \frac{4}{\pi n^2} \cos nt \right) = \sum_{m=0}^{\infty} \left( \frac{2 \sin(2m+1)t}{2m+1} - \frac{4 \cos(2m+1)t}{\pi(2m+1)^2} \right)$$

b) La  $S(t)$  være summen av rekka i a).

$$S(0) = \frac{1}{2} [f(0+) + f(0-)] = \frac{1}{2} [0 + (-\pi)] = -\frac{\pi}{2} \quad (\text{fra grafen i a)}$$

$$S(\pi) = \frac{1}{2} [f(\pi+) + f(\pi-)] = \frac{1}{2} [0 + \pi] = \frac{\pi}{2}$$

$$S(t) = f(t) = f(t - 2\pi) = -\pi - (t - 2\pi) = \pi - t, \quad \pi < t < 2\pi$$

siden  $f(t)$  er periodisk med periode  $2\pi$  og  $f(t) = -\pi - t$  for  $-\pi < t < 0$ .

2 a) Løs

$$y'' + 3y' + 2y = 1 - u(t-1), \quad y(0) = 0, \quad y'(0) = 0.$$

Bruker Laplacetransformasjonen:

$$L\{y\} = Y, \quad s^2 Y + 3sY + 2Y = \frac{1}{s} - \frac{1}{s} e^{-s}$$

$$Y = \frac{1}{s^2 + 3s + 2} \cdot \frac{1}{s} (1 - e^{-s}) = \frac{1}{(s+1)(s+2)s} (1 - e^{-s}) = \left[ \frac{1/2}{s+2} - \frac{1}{s+1} + \frac{1/2}{s} \right] (1 - e^{-s})$$

$$L^{-1} \left\{ \frac{1}{s+1} \right\} = e^{-t}, \quad L^{-1} \left\{ \frac{1}{s+2} \right\} = e^{-2t}, \quad L^{-1} \left\{ \frac{1}{s} \right\} = 1$$

$$y = L^{-1}\{Y\} = \frac{1}{2} e^{-2t} - e^{-t} + \frac{1}{2} - \left[ \frac{1}{2} e^{-2(t-1)} - e^{-(t-1)} + \frac{1}{2} \right] u(t-1)$$

$$y = \begin{cases} \frac{1}{2} e^{-2t} - e^{-t} + \frac{1}{2} & \text{for } t < 1 \\ \frac{1}{2} (1 - e^2) e^{-2t} - (1 - e) e^{-t} & \text{for } t \geq 1 \end{cases}$$

b) Laplacetransformerer ligningen ved å bruke konvolusjonsregelen:

$$y(t) = \cos t + y(t) * \cos t, \quad L\{y\} = Y, \quad L\{\cos t\} = \frac{s}{s^2 + 1}$$

$$Y = \frac{s}{s^2 + 1} + Y \cdot \frac{s}{s^2 + 1}, \quad Y \left( 1 - \frac{s}{s^2 + 1} \right) = \frac{s}{s^2 + 1}, \quad Y \left( \frac{s^2 - s + 1}{s^2 + 1} \right) = \frac{s}{s^2 + 1}$$

$$Y = \frac{s}{s^2 - s + 1} = \frac{s}{(s - 1/2)^2 + 3/4} = \frac{(s - 1/2) + 1/2}{(s - 1/2)^2 + (\sqrt{3}/2)^2}$$

$$y = L^{-1}\{Y\} = e^{\frac{t}{2}} \left( \cos \frac{\sqrt{3}}{2} t + \frac{1}{2} \cdot \frac{1}{\sqrt{3}/2} \sin \frac{\sqrt{3}}{2} t \right) = e^{\frac{t}{2}} \left( \cos \frac{\sqrt{3}}{2} t + \frac{1}{\sqrt{3}} \sin \frac{\sqrt{3}}{2} t \right)$$

3 a) Separasjon av variable:

$$u(x, t) = F(x) \cdot G(t): \quad FG'' - F''G + FG = 0, \quad F'(0) = 0, \quad F'(\pi) = 0$$

$$\frac{G''}{G} + 1 = \frac{F''}{F} = -\lambda \quad (\text{konstant})$$

$$(I) \quad F'' + \lambda F = 0, \quad F'(0) = 0, \quad F'(\pi) = 0, \quad (II) \quad G'' + (\lambda + 1)G = 0$$

$$(I) \quad \lambda < 0, \quad \lambda = -\alpha^2: \quad F(x) = A e^{\alpha x} + B e^{-\alpha x}, \quad F'(x) = \alpha A e^{\alpha x} - \alpha B e^{-\alpha x}$$

$$F'(0) = 0, \quad F'(\pi) = 0 \Rightarrow A = B = 0, \quad F(x) = 0$$

$$\lambda = 0: \quad F(x) = Ax + B, \quad F'(x) = A, \quad F'(0) = F'(\pi) = 0 \Rightarrow A = 0, \quad F_0(x) = 1$$

$$\lambda > 0, \quad \lambda = \alpha^2: \quad F(x) = A \cos \alpha x + B \sin \alpha x, \quad F'(x) = -\alpha A \sin \alpha x + \alpha B \cos \alpha x$$

$$F'(0) = 0 \Rightarrow B = 0, \quad F'(\pi) = 0 \Rightarrow \alpha = n, \quad \lambda = n^2, \quad F_n(x) = \cos nx, \quad n \geq 1$$

$$(II) \quad \lambda = 0: \quad G'' + G = 0, \quad G_0(t) = C_0 \cos t + D_0 \sin t$$

$$\lambda = n^2: \quad G'' + (n^2 + 1)G = 0, \quad G_n(t) = C_n \cos \sqrt{n^2 + 1} t + D_n \sin \sqrt{n^2 + 1} t$$

$$u_0(x, t) = C_0 \cos t + D_0 \sin t$$

$$u_n(x, t) = (C_n \cos \sqrt{n^2 + 1}t + D_n \sin \sqrt{n^2 + 1}t) \cos nx, n \geq 1$$

dvs.  $u_n(x, t) = (C_n \cos \sqrt{n^2 + 1}t + D_n \sin \sqrt{n^2 + 1}t) \cos nx, n = 0, 1, 2, \dots$

b) Superposisjonprinsippet:

$$u(x, t) = \sum_{n=0}^{\infty} (C_n \cos \sqrt{n^2 + 1}t + D_n \sin \sqrt{n^2 + 1}t) \cos nx$$

$$u_t(x, t) = \sum_{n=0}^{\infty} (-C_n \sqrt{n^2 + 1} \sin \sqrt{n^2 + 1}t + D_n \sqrt{n^2 + 1} \cos \sqrt{n^2 + 1}t) \cos nx$$

$$1 + 2 \cos x = u(x, 0) = \sum_{n=0}^{\infty} C_n \cos nx \Rightarrow C_0 = 1, C_1 = 2, C_n = 0 \text{ for } n \geq 2$$

$$\cos 2x = u_t(x, 0) = \sum_{n=0}^{\infty} D_n \sqrt{n^2 + 1} \cos nx \Rightarrow D_2 = 1/\sqrt{5}, D_n = 0 \text{ for } n \neq 2$$

$$u(x, t) = \cos t + 2 \cos \sqrt{2}t \cos x + \frac{1}{\sqrt{5}} \sin \sqrt{5}t \cos 2x$$

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$$p_3(x) = 4 \frac{x(x-1)(x-2)}{(-1)(-2)(-3)} + 1 \frac{(x+1)(x-1)(x-2)}{1(-1)(-2)}$$

$$- 2 \frac{(x+1)x(x-2)}{2 \cdot 1(-1)} + 1 \frac{(x+1)x(x-1)}{3 \cdot 2 \cdot 1}$$

$$= -\frac{2}{3}(x^3 - 3x^2 + 2x) + \frac{1}{2}(x^3 - 2x^2 - x + 2)$$

$$+ (x^3 - x^2 - 2x) + \frac{1}{6}(x^3 - x)$$

$$= x^3 - 4x + 1$$

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$$J = \begin{bmatrix} 2x + y^3 & 3y^2x \\ 6xy & 3x^2 - 3y^2 \end{bmatrix} \quad (2)$$

For  $k = 0, 1, 2$  løs

$$\begin{bmatrix} 2x^{(k)} + (y^{(k)})^3 & 3(y^{(k)})^2x^{(k)} \\ 6x^{(k)}y^{(k)} & 3(x^{(k)})^2 - 3(y^{(k)})^2 \end{bmatrix} \begin{bmatrix} \Delta x^{(k)} \\ \Delta y^{(k)} \end{bmatrix} = - \begin{bmatrix} (x^{(k)})^2 + x^{(k)}(y^{(k)})^3 - 9 \\ 3(x^{(k)})^2y^{(k)} - (y^{(k)})^3 - 4 \end{bmatrix}, \quad (3)$$

$$\begin{aligned} x^{(k+1)} &= x^{(k)} + \Delta x^{(k)} \\ y^{(k+1)} &= y^{(k)} + \Delta y^{(k)} \end{aligned} \quad (4)$$

• Først iterasjon

$$\begin{bmatrix} 18.025 & 22.5 \\ 18.00 & -14.43 \end{bmatrix} \begin{bmatrix} \Delta x^{(0)} \\ \Delta y^{(0)} \end{bmatrix} = \begin{bmatrix} -11.19 \\ 8.825 \end{bmatrix}, \quad (5)$$

og  $\Delta x^{(0)} = 0.05577, x^{(1)} = 1.25577, \Delta y^{(0)} = -0.542009, y^{(1)} = 1.95799.$

• Andre iterasjon

$$\begin{bmatrix} 10.018 & 14.444 \\ 14.753 & -6.7700 \end{bmatrix} \begin{bmatrix} \Delta x^{(1)} \\ \Delta y^{(1)} \end{bmatrix} = \begin{bmatrix} -2.004 \\ 2.2431 \end{bmatrix}, \quad (6)$$

og  $\Delta x^{(1)} = 0.067, x^{(2)} = 1.3228, \Delta y^{(1)} = -0.185, y^{(2)} = 1.7728.$

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a) Vi vil løse

$$u_t = u_{xx}, \text{ for } 0 \leq x \leq 1 \text{ og } t \geq 0,$$

med  $u(x, 0) = \sin(\pi x), 0 \leq x \leq 1$  og  $u(0, t) = u(1, t) = 0, t \geq 0.$

Vi bruke Crank-Nicolson metode:

$$\frac{1}{k}(u(x, t) - u(x, t - k)) = \frac{1}{h^2}(u(x + h, t) - 2u(x, t) + u(x - h, t)). \quad (7)$$

Med  $k = h = 1/4$  og gridpunkter  $x_i = i \cdot h, t_j = j \cdot k$ , la  $u_i^j \approx u(x_i, t_j)$ , og fra

$$\frac{1}{k}(u_i^j - u_i^{j-1}) = \frac{1}{h^2}(u_{i+1}^j - 2u_i^j + u_{i-1}^j).$$

Hvis vi antar at  $u_i^{j-1}$  er kjent, ( $s = \frac{h^2}{k}$ ), får vi

$$-u_{i+1}^j + (2 + s)u_i^j - u_{i-1}^j = su_{i+1}^{j-1}, i = 1, \dots, n-1, u_0^j = u_n^j = 0.$$

For  $k = h = 0.25, s = 0.25, n = 4$

$$\begin{bmatrix} 2.25 & -1 & 0 \\ -1 & 2.25 & -1 \\ 0 & -1 & 2.25 \end{bmatrix} \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = 0.25 \begin{bmatrix} u_1^0 \\ u_2^0 \\ u_3^0 \end{bmatrix} = \begin{bmatrix} 0.17678 \\ 0.25 \\ 0.17678 \end{bmatrix}. \quad (8)$$

Men  $u_1^0 = \sin(\pi \cdot 0.25), u_2^0 = \sin(\pi \cdot 0.5), u_3^0 = \sin(\pi \cdot 0.75).$

b)

$$A = [2.25, -1, 0; -1, 2.25, -1; 0, -1, 2.25];$$

$$b = 0.25 * [\sin(0.25 * \pi); \sin(0.5 * \pi); \sin(0.75 * \pi)];$$

$$u = A \setminus b;$$