Formula Sheet - TMA4130 Mathematics 4N, November 29 2021

Fourier Transform

$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\ \mathrm{d}w$	$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx}\ \mathrm{d}x$
e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/4a}$
$e^{-a\|x\|}$	$\sqrt{\frac{2}{\pi}} \frac{a}{w^2+a^2}$
$\frac{1}{x^2+a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a\|w\|}}{a}$
$\begin{cases} 1 & \text{for } \|x\| < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin wa}{w}$

Fourier series

Let

$$F_N(f) = a_0 + \sum_{n=1}^{N} a_n \cos nx + b_n \sin nx$$

Then

$$\begin{aligned} E_N &= \int_{-\pi}^{\pi} (f(x) - F_N(x))^2 dx \\ &= \int_{-\pi}^{\pi} (f(x))^2 dx - \pi \cdot \left(2a_0^2 + \sum_{n=1}^{N} (a_n^2 + b_n^2) \right) \end{aligned}$$

Laplace Transform

$f(t)$	$F(s) = \int_0^\infty e^{-st} f(t)\ \mathrm{d}t$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
t^n	$\frac{\Gamma(n+1)}{s^{n+1}}$, for $n = 0, 1, 2, \ldots$, $\Gamma(n+1) = n!$
e^{at}	$\frac{1}{s-a}$
$f(t-a)u(t-a)$	$e^{-sa}F(s)$
$\delta(t-a)$	e^{-as}

$$\int x^n \cos ax\ \mathrm{d}x = \frac{1}{a} x^n \sin ax - \frac{n}{a}\int x^{n-1} \sin ax\ \mathrm{d}x$$
$$\int x^n \sin ax\ \mathrm{d}x = -\frac{1}{a} x^n \cos ax + \frac{n}{a}\int x^{n-1} \cos ax\ \mathrm{d}x$$

Trigonometric Identities

$$\begin{aligned}
\cos(\alpha+\beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta \\
\sin\alpha\cos\beta &= \tfrac{1}{2}(\sin(\alpha-\beta) + \sin(\alpha+\beta)) \\
\cos(2\alpha) &= 2\cos^2(\alpha) - 1 \\
\cos(2\alpha) &= 1 - 2\sin^2(\alpha) \\
2\sin\alpha\cos\beta &= \sin(\alpha+\beta) + \sin(\alpha-\beta) \\
2\cos\alpha\sin\beta &= \sin(\alpha+\beta) - \sin(\alpha-\beta)
\end{aligned}$$

Numerics

- Newton's method: $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$.
- Newton's method for system of equations: $\mathbf{x}_{k+1} = \mathbf{x}_k - J(\mathbf{x}_k)^{-1}\mathbf{f}(\mathbf{x}_k)$, with $J = (\partial_j f_i)$.
- Lagrange interpolation: $p_n(x) = \sum_{k=0}^{n} \ell_k(x) f(x_k)$, with $\ell_k(x) = \prod_{j\neq k} \frac{x-x_j}{x_k-x_j}$.
- Interpolation error: $\epsilon_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!} \prod_{k=0}^{n}(x-x_k)$.
- Chebyshev points: $x_k = \cos\left(\frac{2k+1}{2n+2}\pi\right)$, $0 \le k \le n$.
- Newton's divided difference: $f(x) \approx f_0 + (x-x_0)f[x_0,x_1]+$ $(x-x_0)(x-x_1)f[x_0,x_1,x_2] + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1})f[x_0,\ldots,x_n]$, with $f[x_0,\ldots,x_k] = \frac{f[x_1,\ldots x_k]-f[x_0,\ldots,x_{k-1}]}{x_k-x_0}$.
- Trapezoidal rule: $\int_a^b f(x)\,\mathrm{d}x \approx h\left[\frac{1}{2}f(a) + f_1 + f_2 + \cdots + f_{m-1} + \frac{1}{2}f(b)\right]$.
  Error of the trapezoid rule: $|\epsilon| \le \frac{b-a}{12}h^2 \max_{x\in[a,b]} |f''(x)|$.
- Simpson's rule: $\int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + \cdots + 2f_{m-2} + 4f_{m-1} + f_m\right]$.
  Error of the Simpson rule: $|\epsilon| \le \frac{b-a}{180}h^4 \max_{x\in[a,b]} |f^{(4)}(x)|$.
- Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n,\mathbf{y}_n)$.
- Improved Euler (Heun) method: $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{2}h[\mathbf{f}(x_n,\mathbf{y}_n) + \mathbf{f}(x_n+h,\mathbf{y}^*_{n+1})]$, where $\mathbf{y}^*_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_n,\mathbf{y}_n)$.
- Classical Runge–Kutta method: $\mathbf{k}_1 = h\mathbf{f}(x_n,\mathbf{y}_n)$,
  $\mathbf{k}_2 = h\mathbf{f}(x_n+h/2,\mathbf{y}_n+\mathbf{k}_1/2)$,
  $\mathbf{k}_3 = h\mathbf{f}(x_n+h/2,\mathbf{y}_n+\mathbf{k}_2/2)$,
  $\mathbf{k}_4 = h\mathbf{f}(x_n+h,\mathbf{y}_n+\mathbf{k}_3)$,
  $\mathbf{y}_{n+1} = \mathbf{y}_n + \frac{1}{6}\mathbf{k}_1 + \frac{1}{3}\mathbf{k}_2 + \frac{1}{3}\mathbf{k}_3 + \frac{1}{6}\mathbf{k}_4$.
- Backward Euler method: $\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(x_{n+1},\mathbf{y}_{n+1})$.

- Butcher tableaux for different Runge–Kutta methods:

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$

Euler

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

Heun

$$\begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

classical RK

$$\begin{array}{c|c} 1 & 1 \\ \hline & 1 \end{array}$$

backward Euler

- Order conditions:

$p = 1$: $\sum_i b_i = 1$.

$p = 2$: $\sum_i b_i c_i = 1/2$.

$p = 3$: $\sum_i b_i c_i^2 = 1/3$ and $\sum_{i,j} b_i a_{ij} c_j = 1/6$.

$p = 4$: $\sum_i b_i c_i^3 = 1/4$, $\quad \sum_{i,j} b_i c_i a_{ij} c_j = 1/8$, $\quad \sum_{i,j} b_i a_{ij} c_j^2 = 1/12$,
and $\sum_{i,j,k} b_i a_{ij} a_{jk} c_k = 1/24$.

- Finite differences:

$$f'(x) \approx \begin{cases} \dfrac{f(x+h) - f(x)}{h}, & \text{forward difference,} \\ \dfrac{f(x) - f(x-h)}{h}, & \text{backward difference,} \\ \dfrac{f(x+h) - f(x-h)}{2h}, & \text{central difference.} \end{cases}$$

and

$$f''(x) \approx \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

- Crank–Nicolson method for the heat equation: $r = \frac{k}{h^2}$,
$(2 + 2r)u_{i,j+1} - r(u_{i+1,j+1} + u_{i-1,j+1}) = (2 - 2r)u_{ij} + r(u_{i+1,j} + u_{i-1,j})$.