NTNU

Norwegian University of
Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4130 Caculus 4N**

Academic contact during examination: Elisabeth Anna Sophia Köbis and André Massing

Phone: 73412539 (Elisabeth responsible for Problem 1, 2, 3, 7) and 92080635 (Andre responsible for Problem 4, 5, 6, 8, 9)

Examination date: November 29, 2021

Examination time (from–to): 09:00–13:00

Permitted examination support material: C:

- Approved calculator
- One yellow, stamped A5 sheet of self-written notes

Language: English

Number of pages: 9

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave
Originalen er:
1-sidig ☐ **2-sidig** ☒
sort/hvit ☒ **farger** ☐
skal ha flervalgskjema ☐

Date Signature

Problem 1 Laplace transform [10 pts]

a) Consider the function $f\colon [0,\infty) \to \mathbb{R}$ given by

$$f(t) = \begin{cases} 6 \cdot t & \text{for } 0 \le t < 1, \\ 6 & \text{for } t \ge 1. \end{cases}$$

Compute the Laplace transform $\mathcal{L}(f)(s)$ of the function f.

b) Show that for a function $y\colon [0,\infty) \to \mathbb{R}$ whose Laplace transform exists, the following identity holds true:

$$\mathcal{L}\left(\int_0^t \sin(x-t) \cdot y(x)\mathrm{d}x\right)(s) = \frac{Y(s)}{-s^2-1},$$

where $Y(s) = \mathcal{L}(y)(s)$ denotes the Laplace transform of y.

c) Use the results from **a)** and **b)** to compute the solution of the integral equation

$$y(t) + \int_0^t \sin(x-t) \cdot y(x)\mathrm{d}x = f(t)\,.$$

Problem 2 Fourier series [15 pts]

a) Assume that $f\colon \mathbb{R} \to \mathbb{R}$ is a 2π-periodic function and that $a \in \mathbb{N} \setminus \{0\}$ is a constant. Decide for all the following functions whether they are also necessarily periodic. If they are, what is their (fundamental) period?

1. $g_1(x) := f(a \cdot x)$,
2. $g_2(x) := f(x + a)$,
3. $g_3(x) := f(x^a)$,
4. $g_4(x) := a + a \cdot (f(x/a + a))^a$.

b)

1. Calculate the Fourier series of the function $f(x) = |\sin x|$ defined on $[-\pi, \pi]$. Explicitly write down the first five non-vanishing terms of the Fourier sum. **Hint:** You can use the fact that the sine is an odd function, and use your knowledge about Fourier series of even functions.
2. With f from b) 1., how many terms in the partial Fourier sum $F_N(x) = a_0 + \sum_{n=1}^{N}(a_n \cos(nx) + b_n \sin nx)$ need to be taken into account such that the square error

$$E_N = \int_{-\pi}^{\pi} (f(x) - F_N(x))^2 \, dx,$$

is less than 0.01?

Hint: The number is not very large. We recommend that you just calculate (to four digits) the error for the first few partial sums. You can use that $\int_{-\pi}^{\pi} (f(x))^2 \, dx = \pi$.

Problem 3 Heat equation [12 pts]

Consider the following heat equation

$$u_t(x,t) = c^2 u_{xx}(x,t), \quad t \geq 0, x \in [0,\pi]$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \quad t \geq 0$$

and initial condition

$$u(x,0) = \frac{\pi}{2} - \left|x - \frac{\pi}{2}\right|, \quad x \in [0,\pi].$$

a) Show that the Fourier sine series solution of this above heat equation with boundary conditions is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx) e^{-c^2 n^2 t}$$

by using the separation of variables method.

b) Compute the Fourier sine series solution of the above heat equation with the given boundary conditions and initial condition. Write down the three first non-zero terms of the solution.

Problem 4 Wave equation [10 pts]
Consider the wave equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{4}\frac{\partial^2 u}{\partial x^2} = 0$$

on the real line $\mathbb{R}$. Use d'Alembert's solution formula to find the solution $u(x,t)$ satisfying initial conditions

$$\begin{aligned} u(x,0) &= x, \\ \frac{\partial u}{\partial t}(x,0) &= \cos^2(x). \end{aligned}$$

Simplify the resulting expressions as much as possible.

Problem 5 Interpolation [9 pts]

Consider the data points

x_i	-1	1	3
$f(x_i)$	1	4	3

a) Use Lagrange interpolation to find the polynomial of minimal degree interpolating these points. Express the polynomial in the form $p_n(x) = a_n x^n + \cdots + a_1 x + a_0$.

b) Determine the Newton form of the interpolating polynomial.

c) Now add the data point $(x_3, f_3) = (4, 6)$ and compute the resulting interpolation polynomial for the given 4 data points.

Problem 6 Quadrature rules [15 pts]

a) It is known that the quadrature rule $MR[f](a,b)$ defined by the midpoint rule satisfies the error estimate

$$|MR[f](a,b) - \int_a^b f(x)dx| \leqslant \frac{7M}{24}(b-a)^3,$$

where $M = \max_{x\in[a,b]} |f''(x)|$. Which degree of exactness has this quadrature rule and why?

b) Show that the corresponding *composite midpoint rule* $CMR[f](a,b,m)$ defined on m equally spaced subintervals satisfies an estimate for the quadrature error of the form

$$|CMR[f](a,b,m) - \int_a^b f(x)dx| \leqslant M(b-a)\frac{7h^2}{24},$$

where $h = \frac{b-a}{m}$ and M is defined as in **b)**.

c) Consider the the integral $\int_0^1 \cos(x)\,\mathrm{d}x$. Find the number of intervals m which guarantees that the quadrature error for the composite midpoint rule is below 10^{-3}.

d) Write down a `Python` code snippet, which for given function f, interval endpoints a, b and number of intervals m uses the composite midpoint rule to compute the integral $\int_a^b f(x)dx$ numerically.

Problem 7 Nonlinear equations [8 pts]

Let r be a solution of the following equation

$$x + \sin(x - 2) = 0, \quad 0 \leq x \leq 2.$$

Show that the solution is unique by using the intermediate value theorem.
Starting from

$$x_0 = 0.5,$$

perform two iterations of the Newton method.

Problem 8 Numerical methods for ODE [12 pts]

Consider the following implementation of a 3-stage Runge-Kutta method.

```
def rkm(y0, t0, T, f, Nmax):
    ts = [t0]
    ys = [y0]
    dt = (T-t0)/Nmax

    while (ts[-1] < T):
        t, y  = ts[-1], ys[-1]

        k1 = f(t,y)
        k2 = f(t+2/3*dt,y+2/3*dt*k1)
        k3 = f(t+dt,y+dt/2*(k1+k2))

        ys.append(y + dt/4*(k1+3*k2))
        ts.append(t + dt)

    return np.array(ts), np.array(ys)
```

a) Extract the Butcher table from the given implementation. Can you simplify the Butcher table and/or implementation code?

b) Determine the consistency order p of the Runge-Kutta method implemented in **a)**.

c) Now imagine you have run a convergence rate study for three different Runge-Kutta methods, one of which was the method implemented in the code snippet above. You obtained the following tables which tabulate the number of used, equidistant time-steps N against the resulting error.

What are the experimentally observed orders of convergence for each method and which table was likely produced by the method implemented above? Justify your answers!

	N	Error
0	4	0.221199
1	8	0.096199
2	16	0.044258
3	32	0.021231
4	64	0.010403
5	128	0.005141

Table 1

	N	Error
0	4	3.1795e-02
1	8	3.0213e-03
2	16	3.1609e-04
3	32	3.5879e-05
4	64	4.2818e-06
5	128	5.2306e-07

Table 2

	N	Error
0	4	0.071203
1	8	0.010207
2	16	0.001986
3	32	0.000446
4	64	0.000106
5	128	0.000026

Table 3

Problem 9 Numerical Methods for Partial Differential Equations [9 pts]

a) Let $u : [a, b] \to \mathbb{R}$ be a 4 times differentiable function and assume that all derivatives are continuous on $[a, b]$.

Show that the central difference operator

$$\partial^+\partial^- u(x) := \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$$

satisfies

$$u''(x) - \partial^+\partial^- u(x) = \mathcal{O}(h^2) \quad h \to 0.$$

b) On the 2-dimensional unit square $\Omega = (0, 1)^2 \subset \mathbb{R}^2$, consider the two-dimensional Laplace equation

$$\Delta u(x, y) := \partial_{xx} u(x, y) + \partial_{yy} u(x, y) = 0. \tag{1}$$

Assume that $(N + 1)^2$ grid points $\{(x_i, y_j)\}_{i,j}^N$ are defined by uniformly subdividing each axis into N subintervals; that is, for a given double index (i, j), a grid point is given by $(x_i, y_j) = (ih, jh)$ where $h = 1/N$.

Write down the definition of the 5-point stencil used in the finite-difference based discretization of the two-dimensional Laplace operator Δ.

c) On the three-dimensional unit cube $\Omega = (0, 1)^3 \subset \mathbb{R}^3$ consider the three-dimensional Laplace equation

$$\Delta u(x, y, z) := \partial_{xx} u(x, y, z) + \partial_{yy} u(x, y, z) + \partial_{zz} u(x, y, z) = 0. \tag{2}$$

Assume that $(N + 1)^3$ grid points $\{(x_i, y_j, z_k)\}_{i,j,k}^N$ are defined by uniformly subdividing each axis into N subintervals; that is, for a given triple index (i, j, z), a grid point is given by $(x_i, y_j, z_k) = (ih, jh, kh)$ where $h = 1/N$.

Similar to the two-dimensional case, find the corresponding stencil to discretize the three-dimensional Laplace operator Δ using the finite difference method. Give a *short* rationale of how you arrived at the formula.