TMA4125 Matematikk 4N
Spring 2021

Exercise set 1

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The deadline for handing in solutions is Monday 25th January, 12:00.

1 **a)** Let $x \in \mathbb{R}^n$ be a real vector. Show that

$$\lim_{p\to\infty} \|x\|_p = \max_{1\leq i\leq n} |x_i|.$$

You may assume that

- $(\star)$: There is a unique j such that if $i \neq j$ then $|x_i| < |x_j|$.

b) *(Optional)* Show that the statement holds even if $(\star)$ does not hold.

c) The closed unit-disc $\mathbb{D}_p$ with respect to the p-norm $\|\cdot\|_p$ is the set

$$\mathbb{D}_p = \{(x, y) \in \mathbb{R}^2 : \|(x, y)\|_p \leq 1\}.$$

Visualize the disc $\mathbb{D}_p$ for some values of p, at least $p = 1, 2, \infty$.
Hint: Create a dense grid of points $\{(x_i, y_j| \quad i, j = 1, ..., N)\}$, that is a grid with a small spacing. Plot only those points with $\|(x_i, y_j)\|_p \leq 1$. You can calculate the p-norm of a NumPy array `v` by `scipy.linalg.norm(v,p)`. Replace `p` by the value p you want to use. If you want $p = \infty$ you write `numpy.inf`.

2 Recall the Gram-Schmidt orthogonalization process: Say you have a vector space V with a scalar product $\langle\cdot,\cdot\rangle$ and corresponding norm $\|\cdot\|$. For $n \leq \dim(V)$ linearly independent vectors $v_1, ..., v_n$ we can construct a set of orthonormal vectors $u_1, ..., u_n$ by

1. $u_1 = v_1/\|v_1\|$.
2. For $m = 2, ..., n$:

$$w_m = v_m - \sum_{j=1}^{m-1} \langle v_m, u_j\rangle u_j.$$

$$u_m = w_m/\|w_m\|.$$

Consider the vector space $\mathbb{P}^2[0, 1]$ of polynomials with degree ≤ 2, with the scalar product

$$\langle p, q\rangle = \int_0^1 p(x)q(x)\mathrm{d}x$$

and corresponding norm $\|p\| = \langle p, p\rangle^{1/2}$. Starting from the set of vectors $p_1 = 1$, $p_2 = x$, $p_3 = x^2$ use the Gram-Schmidt process to construct an orthonormal set of vectors q_1, q_2, q_3.

Hint: You might find some help from lecture notes in Calculus 3:
https://www.math.ntnu.no/emner/TMA4110/2020h/notater/9-projeksjon.pdf
Note however that they do not normalize their vectors and that they start with vector x instead of 1.

3 Consider the space $L^2[0, 2\pi]$ of square-integrable functions on $[0, 2\pi]$ with the scalar product

$$\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\mathrm{d}x.$$

We define the functions $f_n(x) = \sin(nx)$, $g_m(x) = \cos(mx)$. For integers $m, n \in \mathbb{Z}$ calculate

a) $\langle g_0, g_0\rangle$.

b) $\langle f_n, f_n\rangle, \quad n \neq 0,$

c) $\langle f_m, f_n\rangle, \quad m \neq n,$

d) $\langle f_m, g_n\rangle$.

Hint: Recall the trigonometric identities

$$2\sin(\theta)\sin(\phi) = \cos(\theta - \phi) - \cos(\theta + \phi),$$

$$2\sin(\theta)\cos(\phi) = \sin(\theta + \phi) + \sin(\theta - \phi).$$

4 Consider the space $C[0, 1]$ of continuous functions defined on $[0, 1]$.

a) Show that $\|\cdot\|_C$ defined by

$$\|f\|_C = \max_{x\in[0,1]} |f(x)|$$

is a norm.

b) Define $\|\cdot\|_\star$ by

$$\|f\|_\star = \max_{x\in[0,0.5]} |f(x)|.$$

Is $\|\cdot\|_\star$ a norm? If not, does it satisfy any of the properties of norms?