Lagrange interpolation: Explaining the idea behind the construction of cardinal functions $l_i(x)$. by looking at concrete example.

Given data points:

$\boxed{x_0 \quad x_1 \quad x_2}$
$y_0 \quad y_1 \quad y_2$

distinct values!

1. step: construct a 2nd order polynomial $l_0(x)$ which vanishes at x_1 and x_2:

- $$\tilde{l}_0(x) := (x - x_1)\cdot(x - x_2)$$

Then by construction, we have that $\tilde{l}_0 \in \mathbb{P}_2$,

$$\tilde{l}_0(x_1) = 0$$

$$\tilde{l}_0(x_2) = 0$$

but

$$\tilde{l}_0(x_0) = \tilde{c}_0 = (x_0 - x_1)(x_0 - x_2)$$

2. step. As $\tilde{l}_0(x_0) = \tilde{c}_0$ is not necessarily equal to 1, we simply divide $\tilde{l}_0$ by $\tilde{c}_0$ leading to the definition of l_0:

$$l_0(x) = \frac{\tilde{l}_0(x)}{\tilde{c}_0} = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

Then $l_0(x)$ satisfies $l_0(x_0) = 1$ in addition to $l_0(x_1) = l_0(x_2) = 0$.

Construction of final interpolation polynomial using Lagrange / cardinal functions:

$$p_n(x) := \sum_{i=0}^{n} y_i \, l_i(x)$$

$$p_n(x_0) = y_0 \underbrace{l_0(x_0)}_{1} + y_1 \underbrace{l_1(x_0)}_{0} + \dots + y_n \cdot \underbrace{l_n(x_0)}_{} = y_0$$

$$p_n(x_1) = y_0 \underbrace{l_0(x_1)}_{=0} + y_1 \underbrace{l_1(x_1)}_{=1} + y_2 \underbrace{l_2(x_1)}_{=0} + \dots + y_n \underbrace{l_n(x_1)}_{=0}{}^{0} = y_1$$

In general, we have that

$$p_n(x_j) = \sum_{i=0}^{n} y_i \overbrace{l_i(x_j)}^{\delta_{ij}} = y_j,$$ that is, p_n satisfies the interpolation property.

Newton interpolation

x_0	x_1	x_n	x_{n+1}
y_0	y_1	y_n	y_{n+1}

$$\omega_0(x) = \boxed{1} \in \mathbb{P}^0$$

$\omega_0(x)$

$$\omega_1(x) = \boxed{1} \cdot (x - x_0)$$

$$\omega_2(x) = (x - x_0)\cdot(x - x_1) = \omega_1(x)\cdot(x - x_1)$$

$$\omega_3(x) = (x - x_0)\cdot(x - x_1)\cdot(x - x_2) = \omega_2(x)(x - x_2)$$

$$\vdots$$

$$\omega_n(x) = (x - x_0)(x - x_1) \cdot \quad (x - x_{n-1}) \in \mathbb{P}^n$$

$$\omega_{n+1}(x) = \omega_n(x)\cdot(x - x_n) \in \mathbb{P}^{n+1}$$

$$p_n(x) = c_0 \omega_0(x) + c_1 \omega_1(x) + c_2 \omega_2(x) + \dots + c_n \cdot \omega_n(x)$$

? ? ?

[Compare to Lagrange interpolation $p_n(x) = y_0 l_0(x) + y_1 l_1(x) + \dots\ y_n \cdot l_n(x)$]

$$f(x_0) = y_0 = p_n(x_0) = c_0 \underbrace{\omega_0(x_0)}_{1} + c_1 \underbrace{\omega_1(x_0)}_{0} + c_2 \underbrace{\omega_2(x_0)}_{0} + \dots + c_n \omega_n(x_0)$$

$$= c_0 \cdot 1$$

$$f(x_0) = y_0 = c_0$$

$= y_0 = f(x_0)$ known from previous calculation

$$f(x_1) = y_1 = p_n(x_1) = \boxed{c_0}\underbrace{\omega_0(x_1)}_{=1} + c_1\underbrace{\omega_1(x_1)}_{(x_1 - x_0)} + c_2\underbrace{\omega_2(x_1)}_{0} + \cdots c_n\omega_n(x_1)$$

$$= 1 + c_1 \cdot (x_1 - x_0) + 0 + 0$$

$$\Rightarrow \quad c_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Using divided differences to compute interpolation polynomial in Newton form:

X_i	$Y_i =: f[X_i]$		
0	$\boxed{1 = f[x_0]}$		
$\frac{2}{3}$	$\frac{1}{2} = f[x_1]$	$\boxed{f[x_0,x_1]} = \dfrac{f[x_1]-f[x_0]}{x_1-x_0} = \dfrac{\frac{1}{2}-1}{\frac{2}{3}-0} = \boxed{-\frac{3}{4}}$	
1	$0 = f[x_2]$	$f[x_1,x_2] = \dfrac{f[x_2]-f[x_1]}{x_2-x_1} = \dfrac{0-\frac{1}{2}}{1-\frac{2}{3}} = -\frac{3}{2}$	$\boxed{f[x_0,x_1,x_2] = -\frac{3}{4}}$
2	$1.5 = f[x_3]$	$f[x_2,x_3]$	$f[x_1,x_2,x_3]$

$f[x_0,x_1,x_2,x_3]$

Newton polynomials

$$\omega_0(x) = 1$$

$$\omega_1(x) = (x-x_0) = x$$

$$\omega_2(x) = (x-x_0)(x-x_1) = x\left(x-\frac{2}{3}\right).$$

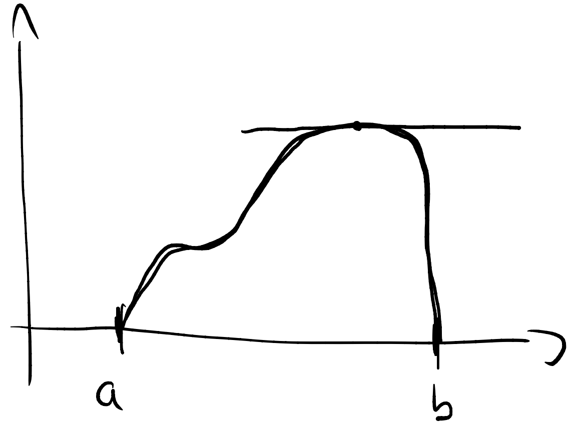
$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0} = \frac{-\frac{3}{2}+\frac{3}{4}}{1} = -\frac{3}{4}$$

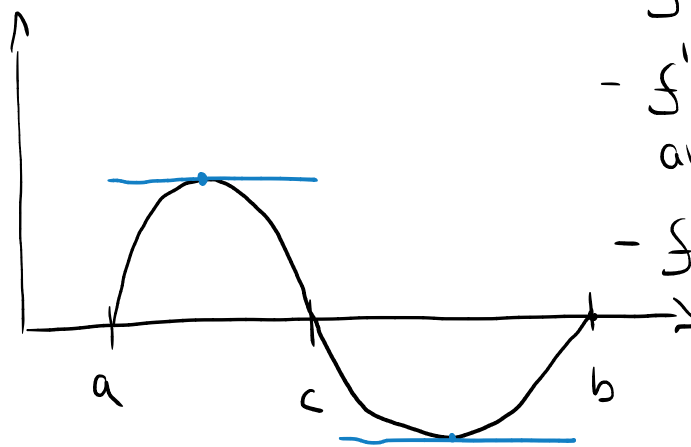
$$\Rightarrow \quad p_2(x) = \underbrace{1}_{f[x_0]} \cdot \underbrace{1}_{\omega_0(x)} \underbrace{\boxed{-\frac{3}{4}}}_{f[x_0,x_1]} \cdot \underbrace{x}_{\omega_1(x)} - \frac{3}{4} \cdot x\left(x-\frac{2}{3}\right).$$

$$p_3(x) = p_2(x) + f[x_0,x_1,x_2,x_3] \cdot \omega_3(x).$$

Rolle' theorem

$f \in C^1([a,b])$
and $f(a) = f(b) = 0$.
Then there is a
$\xi \in [a,b]$ s.t
$f'(\xi) = 0$





We can use Rolle's theorem multiple times:

$f \in C^2[a,b]$:

- f has 3 roots then
- f' has 2 roots, $f' \in C^1$ and
- f'' has 1 root.

Proof of the interpolation error theorem:

- $e(x) := f(x) - p_n(x) \qquad x \in [a,b]$
- Fix this x.
- $\varphi(t) := e(t)\,\omega(x) - e(x)\,\omega(t)$
- $e(x_i) = 0 \qquad i = 0, \dots n$

  $\omega(x_i) = 0$

$$\omega(x) := \omega_{n+1}(x) = (x-x_0)(x-x_1) \dots (x-x_n) = 1 \cdot x^{n+1} + c_n x^n + \dots + c_0$$

$\Rightarrow$ $t = x_i$: $\varphi(x_i) = 0 \qquad i = 0 \dots n$

$t = x \qquad \varphi(x) = 0$

$\Bigg\}$ $\varphi(t)$ has $(n+2)$ distinct zeros | roots !

• $\varphi(t) \in C^{n+1}$ and has $(n+2)$ distinct roots

$\overset{\text{Rolle}}{\Longrightarrow}$

$\varphi'(t) \in C^{n}$ " " $(n+1)$ " "

$\overset{\text{Rolle}}{\Longrightarrow}$

$\varphi''(t) \in C^{n-1}$ n " "

$\Longrightarrow$

$\varphi^{(n+1)}(t) \in C^{0}$ has 1 root

$\exists\ \xi \in (a,b)$ s.t

$$0 = \varphi^{(n+1)}(\xi) = \frac{d^{n+1}}{dt^{n+1}}\bigg|_{t=\xi} \Big(e(t)\,\omega(x) - e(x)\,\omega(t) \Big)$$

$$= \underbrace{\boxed{e^{(n+1)}(\xi)}}_{\mathrm{I}}\,\omega(x) - e(x)\,\underbrace{\boxed{\omega^{(n+1)}(\xi)}}_{\mathrm{II}}.$$

$$\mathrm{I} = \frac{d^{n+1}}{dt^{n+1}}\bigg|_{t=\xi} \Big(f(t) - p_n(t) \Big) = f^{(n+1)}(\xi) - 0$$

$$\mathrm{II} = \frac{d^{n+1}}{dt^{n+1}}\bigg|_{t=\xi} \Big(1\cdot t^{n+1} + c_n t^n + \dots c_0 \Big) = (n+1)! + 0$$

$$0 = \varphi^{(n+1)}(\xi) = f^{n+1}(\xi)\,\omega(x) - e(x)\cdot(n+1)!$$

$$\Rightarrow \quad e(x) = \frac{f^{n+1}(\xi)}{(n+1)!}\,\omega(x) . \qquad \blacksquare$$