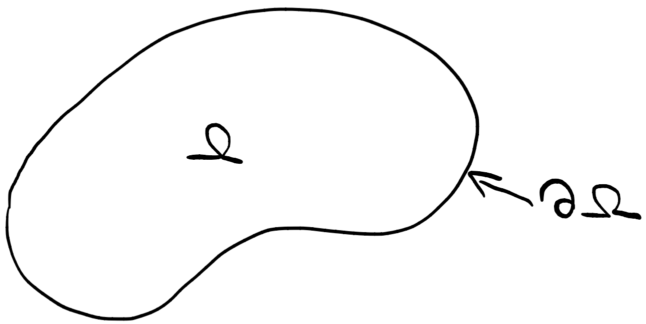
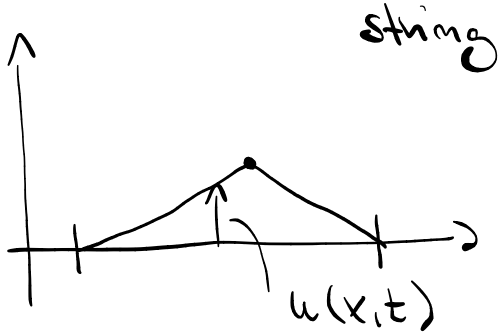
Wave equations

- Laplace operator $\leftarrow \Delta = \partial^2_{x_1} + \partial^2_{x_2} + \quad + \partial^2_{x_d}$
- $c > 0$

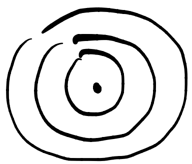
Wave equation:
$$\begin{cases} \partial^2_t u(x,t) - c^2 \Delta u(x,t) = 0 & x \in \Omega \subseteq \mathbb{R}^d \\ & t > 0 \\ \text{boundary conditions} & \\ \text{Dirichlet b.c.}: \quad u(x,t) = u_D(x,t) & x \in \partial\Omega \\ \text{Initial conditions (i.c.)}: \quad u(x,0) = f(x) & x \in \Omega \\ \partial_t u(x,0) = g(x) & x \in \Omega \end{cases}$$



- $d = 1, \quad \Omega = [a,b]$



- $d = 2, 3$



sound

Plan for today:

- Separation of variables to solve the wave equation (finite string)
- d'Alembert's formula: (infinite string).

Separation of variables

Wave equation + b.c. + i.c.

$$\begin{cases} \partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) & x \in [0, L] & \text{Wave equation} & 1) \\ u(0,t) = u(L,t) = 0 & t \geq 0 & \text{boundary condition (b.c.)} & 2) \\ u(x,0) = f(x) & x \in [0, L] & \text{1. initial condition (i.c)} & 3) \\ \partial_t u(x,0) = g(x) & x \in [0,L]. & \text{2. initial condition} & 4) \end{cases}$$

- Separation of variables: Try to find particular solutions $u(x,t) = F(x) G(t)$.
- Plugging this into 1)

$$\left.\begin{aligned} & F(x) G''(t) = c^2 F''(x) G(t) \\ \Rightarrow \quad & \frac{F''(x)}{F(x)} = \frac{G''(t)}{c^2 G(t)} = -k \end{aligned}\right\} \Rightarrow \begin{aligned} & F''(x) + k F(x) = 0 \quad 5) \\ & G''(t) + c^2 k G(t) = 0. \quad 6) \end{aligned}$$

Looking at 5), do a case study for $k<0$, $k=0$, $k>0$.

- $k<0$, $k>0$, the ODE 5) has the following general solution

$$F(x) = \tilde{A} e^{\sqrt{-k}\,x} + \tilde{B} e^{-\sqrt{-k}\,x}.$$

- $k<0$:

$$u(x,t) = F(x)G(t)$$

b.c.: $u(0,t) = u(L,t) = 0 \Rightarrow F(0)G(t) = F(L)G(t) = 0.$

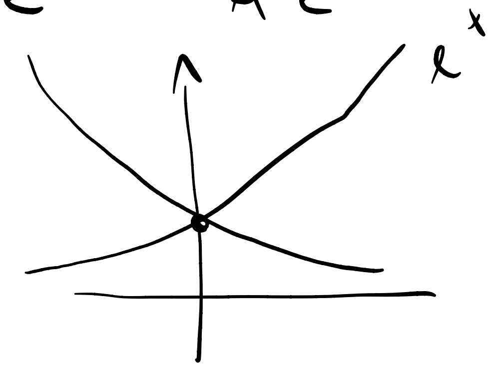
Excluding $\Rightarrow$ trivial solution $G(t) \equiv 0$.

$$F(0) = F(L) = 0.$$

- $F(0) = \tilde{A} + \tilde{B} = 0 \Rightarrow \tilde{A} = -\tilde{B}.$

$$F(L) = \tilde{A} e^{\sqrt{-k}\,L} - \tilde{A} e^{-\sqrt{-k}\,L} = \tilde{A}\left(e^{\sqrt{-k}\,L} - e^{-\sqrt{-k}\,L}\right) = 0$$

- e^{x}, e^{-x}



$\Rightarrow \tilde{A} = 0$

$\Rightarrow$ get only trivial solutions

- $k = 0$ $\Rightarrow$ $F''(x) = 0$ $\Rightarrow$ $F(x) = \tilde{A} + \tilde{B}x$

- b.c. $0 = F(0) = \tilde{A} + \tilde{B}\cdot 0 = \tilde{A}$

$0 = F(L) = \tilde{B}\cdot L \Rightarrow \tilde{B} = 0.$

- $k > 0$: $F(x) = \tilde{A}e^{\sqrt{-k}\,x} + \tilde{B}e^{-\sqrt{-k}\,x}.$

$= \tilde{A}e^{i\sqrt{k}\,x} + \tilde{B}e^{-i\sqrt{k}\,x}$

$= A\cos\sqrt{k}\,x + B\sin\sqrt{k}\,x.$ for some constants A, B.

Euler's formula:
$e^{ix} = \cos x + i\sin x$

- use b.c. to determine A and B.

$0 = F(0) = A$

$0 = F(L) = B\sin\sqrt{k}\,L$ $\overset{\text{looking for a non-trivial solution}}{\Longrightarrow}$ $\sin\sqrt{k}\,L = 0 \Leftrightarrow \sqrt{k}\,L = n\pi$

$\Leftrightarrow k = \left(\frac{n\pi}{L}\right)^2.$

So we have found that

$$F_n(x) = B_n \sin\sqrt{k}\,x = B_n \sin\left(\frac{n\pi}{L}x\right), \qquad n = 1, 2, \ldots$$

satisfies both the wave equation 1) and the boundary conditions 2)

- Let's have a look at eq. 6 :

$$G'' + c^2 k\, G = G''(t) + \left(\frac{c n \pi}{L}\right)^2 G(t) = 0 .$$

$\Rightarrow$ general solution for $G(t)$:

$$G_n(t) = A_n \cos\frac{cn\pi}{L}t + B_n \sin\frac{cn\pi}{L}t$$

- So far the particular solutions

$$u_n(x,t) = F_n(x)\,G_n(t) = \left(A_n \cos\frac{cn\pi}{L}t + B_n \sin\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right)$$

satisfy PDE 1) and b.c. 2). What about i.c.?

. To satisfy i.c. we take a (infinite) linear combination.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi}{L} t + B_n \sin \frac{cn\pi}{L} t \right) \sin\left(\frac{n\pi}{L} x\right)$$

. $f(x) = u(x,0) = \sum_{n=1}^{\infty} \boxed{A_n} \sin\left(\frac{n\pi}{L} x\right)$ $\qquad f: [0,L] \to \mathbb{R}.$

So $\boxed{A_n}$ are thus the Fourier coefficients of the odd extension

$f_o: [-L, L] \to \mathbb{R}.$

$$A_n = \frac{2}{L} \int_0^L f(x) \cdot \sin\left(\frac{n\pi}{L} x\right) dx.$$

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. To determine B_n, we use the second i.c.

$$g(x) = \partial_t u(x,0) = \sum_{n=1}^{\infty} \left(-\frac{cn\pi}{L} \cdot A_n \sin \frac{cn\pi}{L} t + \frac{cn\pi}{L} B_n \cos \frac{cn\pi}{L} t \right) \Bigg|_{t=0} \cdot \sin\left(\frac{n\pi}{L} x\right)$$

$$g(x) = \partial_t u(x,0) = \sum_{n=1}^{\infty} \underbrace{\boxed{\frac{cn\pi}{L} \cdot B_n}}_{\bar{B}_n} \sin \frac{n\pi}{L} x$$

$\Rightarrow$ $\bar{B}_n$ must be the Fourier coefficients of the odd extension of g:

$$\Rightarrow \frac{cn\pi}{L} B_n = \bar{B}_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi}{L} x\right) g(x)\, dx$$

$$\Rightarrow B_n = \frac{2}{cn\pi} \int_0^L \sin\left(\frac{n\pi}{L} x\right) g(x)\, dx. \qquad (8)$$

Theorem: The wave equation 1) - 4) is solved by

$$u(x,t) = \sum_{n=1}^{\infty} \left(A_n \cos \frac{cn\pi}{L} t + B_n \sin \frac{cn\pi}{L} t \right) \sin\left(\frac{n\pi}{L} x\right)$$

where the coefficients A_n, B_n are determined by 7) and 8) respectively.

Wave equation on $\mathbb{R}$ (d'Alembert's formula)

$$\partial_t^2 u(x,t) = c^2 \partial_x^2 u(x,t) \qquad x \in \mathbb{R} \qquad 9)$$

I.C. I $\quad u(x,0) = f(x) \qquad x \in \mathbb{R} \qquad 10)$

$\quad \partial_t u(x,0) = g(x) \qquad x \in \mathbb{R}. \qquad 11)$

- Ansatz: Take two functions $\phi, \psi : \mathbb{R} \to \mathbb{R}$ (2x differentiable), then

$$u(x,t) := \phi(x+ct) + \psi(x-ct) \quad \text{solves } 9):$$

- $\partial_t^2 \phi(x+ct) = c^2 \phi''(x+ct) \quad \Big| \quad \partial_t^2 \psi(x-ct) = c^2 \psi''(x-ct)$
- $c^2 \partial_x^2 \phi(x+ct) = c^2 \phi''(x+ct) \quad \Big| \quad c^2 \partial_x^2 \psi(x-ct) = c^2 \psi''(x-ct).$

use i.c. 10) and 11) to determine ϕ, ψ.

$$\left.\frac{d}{dt}\right|_{t=0}\phi(x+ct) = \left.\phi'(x+ct)\cdot c\right|_{t=0} = \phi'(x)\cdot c$$

- $f(x) = u(x,0) = \phi(x) + \psi(x)$ $\quad +_1)$

$G(x)$ s.t. $G'(x) = g(x)$.

- $g(x) = \partial_t u(x,0) = c\phi'(x) - c\psi'(x)$

$$\Rightarrow \quad \overbrace{c\left(\phi(x) - \psi(x)\right) = \int^x g(x)dx}$$

$$\Rightarrow \quad \phi(x) - \psi(x) = \frac{1}{c}\int^x g(x)dx. \quad +_2)$$

$+_1) + +_2)$: $\phi(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int g(x)dx\right)$

$+_1) - +_2)$: $\psi(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int g(x)dx\right)$

$$u(x,t) = \phi(x+ct) + \psi(x-ct) = \frac{1}{2}f(x+ct) + \frac{1}{2c}\int^{x+ct} g(x)dx + \frac{1}{2}f(x-ct) - \frac{1}{2c}\int^{x-ct} g(x)dx$$

$$= \int_{x-ct}^{x+ct} g(s)ds.$$

$$\Rightarrow \boxed{u(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\cdot\int_{x-ct}^{x+ct} g(s)ds}$$

d'Alembert' formula.

Theorem (d'Alembert' formula).

The wave equation 9) - 11) is solved by

$$u(x,t) = \frac{1}{2}\Big(f(x+ct) + f(x-ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds .$$

Note: There was a typo in the last equation, during the lecture I accidently wrote $\frac{1}{2}\big(f(x+ct) - f(x-ct)\big)$ instead of

$$\frac{1}{2}\big(f(x+ct) + f(x-ct)\big).$$