Exercise #13

15. November 2022

This exercise should only be handed in, if you have (or expect to have) precisely 7 approved exercises amongst the first 12. The deadline is Friday, November 25, 16:00. You won't get any detailed feedback about your solutions, and there won't be any guidance for this exercise sheet in the exercise classes, but we will publish solutions.

Problem 1. (Ralston's method)

Ralston's method is given by the following Butcher tableau:

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 3/4 & 0 & 3/4 & 0 \\ \hline & 2/9 & 1/3 & 4/9 \end{array}$$

a) Determine the order of this Runge–Kutta method.

b) Consider the initial value problem $y'(t) = te^{-y}$, $y(0) = 0$. Using a time-step size $h = 0.48$, use Ralston's method to compute the *first step*, by hand. Make sure to also write down all stage derivatives k_1, k_2 and k_3.

Problem 2. (Adaptivity)

Consider the following implementation of a Runge–Kutta method:

```
import numpy as np
import matplotlib.pyplot as plt

T = 2.0
y0 = 0.0
t = 0.0
h = 0.25

def f(t,y):
    return  t*np.exp(-y)

ys = [y0]
ts = [t]

while(t+h < T):
    t, y = ts[-1], ys[-1]
    k1 = f(t,y)
    k2 = f(t+h/2, y + h*k1/2)
    k3 = f(t+3*h/4, y + 3*h*k2/4)
    k4 = f(t+h, y + h*(2*k1 + 3*k2 + 4*k3)/9)
    y = y + h*(7*k1/24 + k2/4 + k3/3 + k4/8)
    ys.append(y)
    ts.append(t + h)

plt.plot(ts, ys, 's-')
```

a) Based on the implementation above, write down the Butcher tableau for this RK method.

b) If we run the code, how many time steps will be computed?

c) Based on the tableau, determine the order of this method.

d) Considering the same initial value problem and time-step size as in Problem 1, compute the first time step using the method implemented above.
Hint: before you actually compute the stage derivatives k_i, check whether you can re-use any of the calculations done for Problem 1.

e) Consider now the combination of this method with the one from Problem 1, to create an adaptive scheme. Remember the error estimate:

$$\hat{\epsilon}_{n+1} = \left|y_{n+1} - y^*_{n+1}\right| = h\left|\sum_{i=1}^{s}(b_i - b_i^*)k_i\right| ,$$

with the superscript $*$ referring to the lowest-order method among the two. Based on the calculations done so far, compute $\hat{\epsilon}_1$.

f) Comparing $\hat{\epsilon}_1$ with the tolerance tol $= 0.001$, check if the first step we computed is acceptable and, if not, compute the new time-step size h_{new} based on the formula developed in class:

$$h_{\text{new}} = P\left(\frac{\text{tol}}{\hat{\epsilon}_1}\right)^{\frac{1}{p+1}} h ,$$

with $P = 0.9$.

Problem 3. (Stability)

For an ODE $y'(t) = f(t, y)$, the so-called *explicit midpoint method* is given by

$$y_{n+1} = y_n + hf(t_n + 0.5h,\ y_n + 0.5hf(t_n, y_n)) .$$

Consider, in particular, the linear autonomous equation where $f(y) = \lambda y$, with $\lambda \in \mathbb{C}$.

a) Write down the Butcher tableau for this Runge–Kutta method.

b) Determine the stability function $R(z)$ of the explicit midpoint method, that is, the function that allows us to write $y_{n+1} = [R(h\lambda)]\, y_n$.

c) Determine the stability region $\mathcal{S}$.

d) For $\lambda = -20$, what is the interval of time-step sizes h for which we get a stable solution?

Problem 4.

For an ODE $y'(t) = f(t, y)$, the *implicit midpoint method* is given by

$$y_{n+1} = y_n + hf(t_n + 0.5h,\ 0.5(y_n + y_{n+1}))\,.$$

a) Write down the Butcher tableau for the implicit midpoint method and find its order.

b) Determine the stability function $R(z)$ of the midpoint method and show that this method is A-stable.

Problem 5.

We want to compute a numerical solution of the initial value problem

$$y'(t) = -y(t)^2 - y(t), \qquad y(0) = 1,$$

using the implicit Euler method with step size $h > 0$.

a) Write down an explicit formula for the step y_{n+1} given y_n.

   NB: In each step you have to solve a quadratic equation that has two different solutions. However, only one of these solutions makes sense. Part of the problem is to choose the correct one!

b) Show that the steps y_n satisfy $\lim_{k\to\infty} y_n = 0$.

   Hint: Use the fixed point theorem!

Problem 6.

We want to solve the second order ODE

$$y'' = -11y' - 10y + 2, \qquad y(0) = 1,\ \ y'(0) = -1,$$

using the explicit Euler method.

What is the largest step length for which we obtain a stable numerical solution?