Exercise #1

22. August 2022

Exercises marked with a (J) should be handed in as a Jupyter notebook.

Optional exercises will not be corrected.

Problem 1. (Taylor Polynomials)

a) Compute all Taylor polynomials of $f(x) = x^4 + 2x^3 + x^2 + 5$ around $x_0 = 1$

b) Compute the Taylor series of $g(x) = \ln(1 + x)$ around $x_0 = 0$

Problem 2. (Numerical differentiation)

Let

$$u'(x) = \frac{3u(x) - 4u(x-h) + u(x-2h)}{2h} + e(h).$$

a) Find an expression for the error $e(h)$.

b) Let $u(x) = x\cos(x)$. Use the expression above to find approximations to $u'(x)$ at $x = \pi/2$, using $h = 0.1$. How large is the error $|e(h)|$ in this case?

c) Verify your results numerically as described in Preliminaries, section 3.2. Use $h = 0.1 \cdot 2^{-i}$, $i = 0, 1, \cdots, 6$. Present your result as a convergence plot.

Problem 3. (Numerical differentiation)

We want to find a difference formula of the form

$$u''(x) \approx \frac{1}{h^2}\left(a_1 u(x) + a_2 u(x-h) + a_3 u(x - \frac{1}{2}h)\right),$$

where a_1, a_2 and a_3 are constants to be determined.

Find the coefficients a_1, a_2 and a_3 which makes this scheme convergent. Find an expression for the error term.

Problem 4. (Boundary value problem)

Given the two point boundary value problem:

$$u_{xx} + 2u_x + \pi^2 u = \cos(\pi x) - \pi(x+1)\sin(\pi x), \quad 0 \le x \le 2, \qquad u(0) = 0 \qquad u(2) = 1$$

a) Verify that the exact solution is

$$u(x) = \frac{x}{2}\cos(\pi x).$$

b) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = 2/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \dots, N$.

c) Let $N = 4$ and use the above formula to find approximations $U_i \approx u(x_i)$, $i = 1, 2, 3$. (That is: Set up the system of equations, and solve it). Compare with the exact solution.

d) (J) Modify the code `Example 1, BVP` in the note on boundary value problems, and solve the problem numerically. Use $N = 10, 20, 40$ in your simulation. For each N, write down the error

$$e(h) = \max_{i=0,\dots,N} |u(x_i) - U_i|.$$

What can you deduce about the order of the scheme from this experiment?

The next two exercises are optional and should not be handed in

Problem 5. (Boundary value problem)

Given the two point boundary value problem:

$$u_{xx} - \frac{2}{x}u_x + \frac{2}{x^2}u = -\frac{x\pi^2}{2}\cos(\pi x), \quad 1 \le x \le 2, \qquad u(1) = -\frac{1}{2} \qquad u(2) = 1$$

a) Verify that the exact solution is

$$u(x) = \frac{x}{2}\cos(\pi x).$$

b) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = 1/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \dots, N$.

c) Let $N = 4$ and use the above formula to find approximations $U_i \approx u(x_i)$, $i = 1, 2, 3$. (That is: Set up the system of equations, and solve it). Compare with the exact solution.

d) (J) Modify the code `Example 1, BVP` in the note on boundary value problems, and solve the problem numerically. Use $N = 10, 20, 40$ in your simulation. For each N, write down the error

$$e(h) = \max_{i=0,\dots,N} |u(x_i) - U_i|.$$

What can you deduce about the order of the scheme from this experiment?

Problem 6. (Boundary value problem - a nonlinear problem)

Given the two point boundary value problem:

$$0.1u'' + u' + u^2 = 0, \quad u(0) = 1, \quad u(1) = 0.$$

a) Set up a finite difference scheme for this problem, using central differences. Use $\Delta x = 1/N$ as the grid size, and let $x_i = i\Delta x$, $i = 0, 1, \dots, N$. You end up with a system of nonlinear equations.

b) (J). Solve the problem in python. Use $N = 50$ (although you may try other values as well).

Hint: Use the python function `scipy.optimize.fsolve`. Read the documentation to figure out how to use it.