Exercise #2

26. August 2022

Optional exercises will not be corrected.

Problem 1. (Polynomial interpolation)

You are allowed (but not required) to use python for this problem. Approximate answers obtained by numerical methods are accepted. Consider the function

$$f(x) = 2^{-x^2+3x-2}.$$

a) Interpolate $f(x)$ by a polynomial of minimal degree, by using the interpolation points

$$x_0 = -1, \qquad x_1 = 0, \qquad x_2 = 1, \qquad x_3 = 2.$$

What is the maximal error on the intervals $[-1, 2]$ and $[-10, 10]$? Plot $f(x)$ and the interpolating polynomial in the same plot, on the interval $[-1, 2]$.

b) The *Chebyshev nodes* on an interval $[a, b]$ are defined as

$$\tilde{x}_i = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{(2i+1)\pi}{2(n+1)}\right), \qquad i = 0, \dots, n.$$

Find the Chebyshev nodes on the interval $[-1, 2]$ for $n = 3$.
(If you are interested in more information about Chebyshev interpolation, you can have a look at the note "Interpolation", Section 2.3.)

c) Interpolate $f(x)$ in the Chebyshev nodes for $n = 3$. What is the maximal error on the interval $[-1, 2]$? Plot $f(x)$ and the interpolating polynomial in the same plot, on the interval $[-1, 2]$.

d) Plot the error as a function in x, for both the interpolation in a) and the interpolation in c), on the interval $[-1, 2]$. Plot both errors in the same plot.

Problem 2. (Lagrange interpolation)

In the lectures and in Maths 3, you have seen, how interpolation polynomials can be constructed by solving a system of linear equations. In this exercise, we will discuss an alternative, direct approach, called *Lagrange interpolation.*
(See the note "Interpolation", Section 2.1 for more information about that approach.)

Given $n+1$ distinct points x_i, $i = 0, \dots, n$, we define the associated *cardinal functions* as

$$\ell_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j} = \frac{x - x_0}{x_i - x_0} \cdots \frac{x - x_{i-1}}{x_i - x_{i-1}} \cdot \frac{x - x_{i+1}}{x_i - x_{i+1}} \cdots \frac{x - x_n}{x_i - x_n}, \qquad i = 0, \dots, n.$$

a) Compute the cardinal functions $\ell_i(x)$ explicitly for the points $x_0 = 0$, $x_1 = 1$, $x_2 = 4$.

b) Verify that the cardinal functions satisfy the equations

$$\begin{aligned} \ell_i(x_i) &= 1 \quad \text{for all } i = 0, \dots, n, \\ \ell_i(x_j) &= 0 \quad \text{for all } i, j = 0, \dots, n \text{ with } i \neq j. \end{aligned}$$

c) Assume we are given points $(x_i, y_i)_{i=0}^{n}$, with distinct values x_i. Show that the interpolation polynomial p_n is given by

$$p_n(x) = \sum_{i=0}^{n} y_i \ell_i(x).$$

(Hint: Verify that each ℓ_i is a polynomial of degree n and that the interpolation condition $p_n(x_i) = y_i$ is satisfied.)

Problem 3. (Trapezoidal rule)

We consider in this problem the numerical integration of the function $f(x) = \sqrt{x}$ on the interval $I = [0, 1]$.

a) Implement the composite trapezoidal rule for the numerical computation of this integral.

b) Estimate the numerical converge rate for this method. That is, assume that the error satisfies

$$\mathrm{err}_m := \left| T_m(0,1) - \int_0^1 \sqrt{x}\, dx \right| \approx Ch^p,$$

and estimate numerically the number p. *(Your estimated p is not necessarily an integer!)*

c) Explain why your result from part b) does not contradict the usual theory of the trapezoidal rule.

Problem 4. (Gauß–Legendre quadrature)

The Gauß–Legendre quadrature $G(f)(-1,1)$ with 3 nodes on the interval $[-1,1]$ is given by the nodes

$$x_0 = -\sqrt{\frac{3}{5}}, \qquad x_1 = 0, \qquad x_2 = +\sqrt{\frac{3}{5}},$$

and the weights

$$w_0 = \frac{5}{9}, \qquad w_1 = \frac{8}{9}, \qquad w_2 = \frac{5}{9}.$$

a) Transfer this quadrature rule to an arbitrary interval (a,b) to obtain an approximation

$$G(f)(a,b) \approx \int_a^b f(x)\, dx.$$

One can show that the error for this quadrature rule satisfies

$$E(a,b) = \int_a^b f(x)\, dx - G(f)(a,b) = \frac{(b-a)^7}{2016000} f^{(6)}(\xi) \tag{1}$$

for some $\xi \in (a,b)$, provided the function f is 6-times continuously differentiable.

b) Based on the expression (1), what error estimates would you expect for the composite Gauß–Legendre rule G_m? Give a brief explanation.

c) Assume that (a,b) is chosen small such that $f^{(6)}$ is almost constant on (a,b). Derive error estimates for

$$E_1(a,b) = \int_a^b f(x)\, dx - G_1(a,b) \qquad \text{and} \qquad E_2(a,b) = \int_a^b f(x)\, dx - G_2(a,b),$$

only based on the values of $G_1(a,b)$ and $G_2(a,b)$.

The next exercises are optional and should not be handed in!

Problem 5. (Inverse interpolation)

We are given a function $f(x)$ with a root r in an interval $[a, b]$, that is, $f(r) = 0$. Moreover, we assume that the interval is chosen sufficiently small such that the function f is either increasing on the whole interval, or decreasing on the whole interval. We want to find r by using interpolation. To that end, we compute an interpolating function $p(y)$ through the points $(f(x_i), x_i)_{i=0}^n$, for nodes $x_i \in [a, b]$. Then $p(0)$ should be an approximation of r.

a) Let $f(x) = x^3 - 7$ and $[a, b] = [1.5, 2.0]$. Choose the nodes $x_i = (1.5, 1.75, 2.0)$ and compute an interpolation polynomial in order to approximate the root r. How close are you to the actual root?

b) Repeat the example with $n + 1$ equidistant nodes over the interval $[a, b]$. Choose $n = 2$, 4, 8, and 16, find the approximation in each case as well as the error.

c) Repeat the example with spline interpolation.

Problem 6. (Midpoint rule)

We consider in this problem the numerical integration of the function $f(x) = \log(x)$ on the interval $I = [0, 1]$.

a) Implement the composite midpoint rule for the numerical computation of this integral.

b) Estimate the numerical converge rate for this method. That is, assume that the error satisfies

$$\mathrm{err}_m := \left| M_m(0,1) - \int_0^1 \log(x)\,dx \right| \approx Ch^p,$$

and estimate numerically the number p.

Explain why your result from part b) does not contradict the usual theory of the midpoint rule.