Exercise #6

26. September 2022

Problem 1.

Find the Fourier transform of the following functions $f : \mathbb{R} \to \mathbb{R}$:

a) The function

$$f(x) = \mathrm{e}^{-|x|}.$$

b) The function

$$f(x) = x^2 \mathrm{e}^{-x^2}.$$

Hint: Differentiate e^{-x^2} twice, and use the table in Kreyszig, p. 536.

Problem 2.

Define the functions $f, g : \mathbb{R} \to \mathbb{R}$ as

$$f(x) := e^{-x^2} \quad \text{and} \quad g(x) := e^{-2x^2}.$$

Use the Fourier transform in order to compute the convolution $f * g$.

You may (and should) use the table in Kreyszig, p. 536, for the computation of the Fourier transform of e^{-ax^2}.

Problem 3.

Define the functions $f, g\colon \mathbb{R} \to \mathbb{R}$ as

$$f(x) := \begin{cases} 1 & \text{if } 0 < x < 1, \\ 0 & \text{else,} \end{cases} \qquad \text{and} \qquad g(x) := \begin{cases} x & \text{if } 0 < x \leq 1, \\ 2 - x & \text{if } 1 < x < 2, \\ 0 & \text{else.} \end{cases}$$

a) Use the definition of convolution in order to show that $f * f = g$.

b) Compute the Fourier transform of f.

c) Compute the Fourier transform of g.

Problem 4.

a) Compute explicitly the Fourier matrices $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_4$.

b) Compute by hand the discrete Fourier transform of the vector $f = (1, 2, 3, 1)$.

c) Assume that the discrete Fourier transform of the vector $f \in \mathbb{R}^{11}$ is given as

$$\hat{f} = (0, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0).$$

Express the entries f_n of f in terms of sine and cosine functions.

The next exercises are optional and should not be handed in!

Problem 5.

Compute the convolution $f * f$, where $f\colon \mathbb{R} \to \mathbb{R}$ is the function

$$f(x) = \operatorname{sinc}(x) := \begin{cases} \dfrac{\sin(x)}{x} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

Hint: Use the Fourier transform!

Problem 6.

Compute the Fourier transform of the following functions $f\colon \mathbb{R} \to \mathbb{R}$:

a) The function

$$f(x) = xe^{-|x|}.$$

b) The function

$$f(x) = \begin{cases} \sin(x) & \text{if } 0 < x < \pi, \\ 0 & \text{else.} \end{cases}$$

Problem 7. (See Problem 7 in the exam from spring 2022)

a) Compute by hand the discrete Fourier transform of the vector $f = (1/2, 1, 1/2, 0)$.

b) Let $c \in \mathbb{R}$ be given, and assume that, for some signal $g \in \mathbb{R}^4$ we obtain $\hat{g} = (0, c, 0, c)$. What is the simplest function $g(x)$ that could have been sampled?

c) Is the inverse Fourier transform $h = \mathcal{F}_8^{-1}\hat{h}$ of the vector $\hat{h} = (0, 0, 0, 0, 0, 0, 1, 0)$ real-valued?