# Exercise #8

## 10. October 2022

**Problem 1.** (Finite difference method for the wave equation)
Consider the wave equation

$$
\begin{aligned}
u_{tt} &= 4u_{xx}, & -5 \le x \le 5, \quad t \ge 0, \qquad (1a)\\
u(-5,t) &= u(5,t) = 0, & t \ge 0, \qquad (1b)\\
u(x,0) &= f(x), \qquad u_t(x,0) = g(x), & -5 \le x \le 5. \qquad (1c)
\end{aligned}
$$

a) Find the exact solutions of the wave equation for the following two sets of initial conditions:

- The conditions
$$f(x) = \cos\left(\frac{\pi x}{2}\right) \quad \text{and} \quad g(x) = 0. \tag{2}$$

- The conditions
$$f(x) = 0 \quad \text{and} \quad g(x) = \pi\cos\left(\frac{\pi x}{2}\right). \tag{3}$$

b)
- Set up a finite difference scheme, using central differences for the second derivatives in both directions. Use step sizes $h$ and $k$ in the spatial and temporal directions respectively. Notice that special care has to be taken for the first temporal step. The final scheme should be written in the form

$$
\begin{aligned}
U_i^1 &= \cdots, & i &= 1,\dots,M-1,\\
U_i^{n+1} &= \cdots, & i &= 1,\dots,M-1, \qquad n = 1,2,\dots,
\end{aligned}
$$

where the $\cdots$ consist of given and/or previously computed values.

NB! Your solution should include an explanation on how the explicit scheme is obtained.

- Use the initial values (2) Let $h = 1$ and $k = 0.5$ and use the formulas above to find approximations to $u(x_i, 0.5)$, $i = 1, \ldots, 9$ (by hand!).

c) Implement and test the scheme derived in point (b). A template for the implementation can be found in the Jupyternote `PDE_wave_fdm.ipynb`.

It is more interesting to study the animations of the simulations, however, for the hand-in, you are asked to plot the solutions for $t = 0, 0.4, 1.2$ and 2 if nothing else is said.

- Use the initial values from (2). Let $h = 0.1$ and $k = 0.05$. Plot the solutions, and compare with the exact solution.

- Repeat the experiment from point (d) with temporal step sizes $k = 0.025$ and then with $k = 0.1$, without changing $h$. What do you observe in these cases?

  You may also like to try (but not hand in) with $k$ just a tiny bit larger than 0.05, and observe what happens in this case. You should observe instabilites, but you may have to increase $t_{end}$ to observe it.

**The next bullets points are optional and should not be handed in**

- Repeat the experiment above with the initial values given by (3).

- Use the initial values (2), but let now the wave speed $c$ be given by

$$c(x) = \begin{cases} 1 & \text{for } x \leq 0, \\ 2 & \text{for } x > 0. \end{cases}$$

  Plot the solution for $t = 0, 0.4, 1.2$ and 2.

**Problem 2.**

Use D'Alembert's method to find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

for $t > 0$ and all $x \in \mathbb{R}$ with initial conditions

$$u(x,0) = \cos x \qquad \text{and} \qquad \frac{\partial u(x,0)}{\partial t} = xe^{-x^2}.$$

**Problem 3.** (1D Heat Equation)

Solve the differential equation

$$\begin{cases} \dfrac{\partial}{\partial t}u - 2\dfrac{\partial^2}{\partial x^2}u = 0, \\ u(0,t) = u(\pi,t) = 0, \\ u(x,0) = x(x-\pi) \end{cases}$$

for $x \in [0,\pi]$ and $t \in [0,\infty]$.

**The next exercises are optional and should not be handed in!**

**Problem 4.**

Use d'Alembert's method in order to find the solution of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \text{for } x > 0 \text{ and } t > 0$$

with initial conditions

$$u(x,0) = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = 0 \qquad \text{for all } x > 0$$

and boundary condition

$$u(0,t) = \sin(t) \qquad \text{for all } t > 0.$$

**Problem 5.**

Vibrations in a stiff metal rod can be roughly described by the fourth order PDE

$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}.$$

We assume now that the metal rod is simply supported at both ends at $x = 0$ and $x = \pi$, and has zero curvature at both ends. This corresponds to the boundary conditions

$$u(0,t) = u(\pi,t) = 0 \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2}(0,t) = \frac{\partial^2 u}{\partial x^2}(\pi,t) = 0$$

for all $t > 0$.

a) We use the idea of separation of variables and consider solutions of the form $u(x,t) = F(x)G(t)$. Derive ordinary differential equations for the functions $F$ and $G$.

b) Verify that all functions of the form

$$F(x) = A\sin(\beta x) + B\cos(\beta x) + C\sinh(\beta x) + D\cosh(\beta x)$$

with $\beta > 0$ satisfy the ODE for $F$ derived in the previous point. For which values of $\beta > 0$ are the boundary conditions satisfied? What are the corresponding solutions of the equation for $G$?