10.8

Evaluation of Integrals

Using (5), (11), or (13), show that the given integrals represent the indicated functions. (Can you see that the integral tells you which formula to use? Show the details of your work.)

1. $$\int_0^\infty \frac{\cos xw + w \sin xw}{1 + w^2}\, dw = \begin{cases} 0 & \text{if } x < 0 \\ \pi/2 & \text{if } x = 0 \\ \pi e^{-x} & \text{if } x > 0 \end{cases}$$

2. $$\int_0^\infty \frac{\sin w \cos xw}{w}\, dw = \begin{cases} \pi/2 & \text{if } 0 \leqq x < 1 \\ \pi/4 & \text{if } x = 1 \\ 0 & \text{if } x > 1 \end{cases}$$

3. $$\int_0^\infty \frac{1 - \cos \pi w}{w} \sin xw\, dw = \begin{cases} \pi/2 & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$$

4. $$\int_0^\infty \frac{\cos (\pi w/2) \cos xw}{1 - w^2}\, dw = \begin{cases} (\pi/2) \cos x & \text{if } |x| < \pi/2 \\ 0 & \text{if } |x| > \pi/2 \end{cases}$$

5. $$\int_0^\infty \frac{\cos xw}{1 + w^2}\, dw = \frac{\pi}{2} e^{-x} \quad \text{if } x > 0$$

6. $$\int_0^\infty \frac{w^3 \sin xw}{w^4 + 4}\, dw = \frac{\pi}{2} e^{-x} \cos x \quad \text{if } x > 0$$

Fourier Cosine Integral Representation

Represent the following functions $f(x)$ in the form (11).

7. $f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

8. $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

9. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

10. $f(x) = \begin{cases} a^2 - x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

11. $f(x) = 1/(1 + x^2)$ $[x > 0$, see (15)]

12. $f(x) = e^{-x} + e^{-2x}$ $(x > 0)$

Fourier Sine Integral Representation

Represent the following functions $f(x)$ in the form (13).

13. $f(x) = \begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

15. $f(x) = \begin{cases} \sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

16. $f(x) = \begin{cases} \pi - x & \text{if } 0 < x < \pi \\ 0 & \text{if } x > \pi \end{cases}$

17. $f(x) = \begin{cases} e^x & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

18. $f(x) = \begin{cases} e^{-x} & \text{if } 0 < x < 1 \\ 0 & \text{if } x > 1 \end{cases}$

19. **(CAS. Sine integral)** Plot Si(u) for positive u. Does the sequence of the maximum and minimum values make the impression that it converges and has the limit $\pi/2$? Investigate the Gibbs phenomenon graphically.

20. **PROJECT. Properties of Fourier Integrals. (a) Fourier Cosine Integral.** Show that (11) implies

(a1) $$f(ax) = \frac{1}{a} \int_0^\infty A\left(\frac{w}{a}\right) \cos xw\, dw \quad (a > 0)$$

(a2) $$xf(x) = \int_0^\infty B^*(w) \sin xw\, dw, \quad B^* = -\frac{dA}{dw}, \quad A \text{ as in (10)}$$

(a3) $$x^2 f(x) = \int_0^\infty A^*(w) \cos xw\, dw, \quad A^* = -\frac{d^2A}{dw^2}.$$

(b) Solve Prob. 8 by applying (a3) to the result of Prob. 7.

(c) Verify (a2) for $f(x) = 1$ if $0 < x < a$ and $f(x) = 0$ if $x > a$.

(d) Fourier Sine Integral. Find formulas for the Fourier sine integral similar to those in (a).

PROBLEM SET 10.10

10.10

Calculation of Fourier Transforms

Find the Fourier transforms of the following functions $f(x)$ (without using Table III, Sec. 10.11). Show the details of your work.

1. $f(x) = \begin{cases} 1 & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$

2. $f(x) = \begin{cases} e^{-kx} & \text{if } x > 0 \quad (k > 0) \\ 0 & \text{if } x < 0 \end{cases}$

3. $f(x) = \begin{cases} e^{x} & \text{if } -a < x < a \\ 0 & \text{otherwise} \end{cases}$

4. $f(x) = \begin{cases} e^{kx} & \text{if } x < 0 \quad (k > 0) \\ 0 & \text{if } x > 0 \end{cases}$

5. $f(x) = \begin{cases} x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

6. $f(x) = \begin{cases} x^2 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

7. $f(x) = \begin{cases} xe^{-x} & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$

8. $f(x) = \begin{cases} e^{x} & \text{if } x < 0 \\ e^{-x} & \text{if } x > 0 \end{cases}$

9. $f(x) = \begin{cases} -1 & \text{if } -a < x < 0 \\ 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$

10. $f(x) = \begin{cases} |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise} \end{cases}$

Use of Table III in Sec. 10.11

11. Obtain $\mathcal{F}(e^{-x^2/2})$ from formula 9 in Table III.
12. Solve Prob. 7 by (9) in the text and formula 5 in Table III.
13. In Table III obtain formula 7 from formula 8.
14. Solve Prob. 8 by formula 5 in Table III.
15. **(Convolution)** Solve Prob. 7 by convolution.
16. **TEAM PROJECT. Shifting.** (a) Show that if $f(x)$ has a Fourier transform, so does $f(x - a)$, and $\mathcal{F}\{f(x - a)\} = e^{-iwa}\mathcal{F}\{f(x)\}$.
    (b) Using (a), obtain formula 1 in Table III, Sec. 10.11, from formula 2.
    (c) *Shifting on the w-axis.* Show that if $\hat{f}(w)$ is the Fourier transform of $f(x)$, then $\hat{f}(w - a)$ is the Fourier transform of $e^{iax}f(x)$.
    (d) Using (c), obtain formula 7 in Table III from formula 1, and formula 8 from formula 2.

PROBLEM SET 11.1

11.1

1. (Fundamental Theorem) Prove Fundamental Theorem 1 for second-order differential equations in two and three independent variables.

Verification of Solutions

In each case verify that the given function is a solution of the indicated equation and sketch or plot a figure of the solution as a surface in space.

Wave Equation (1) (with suitable c)

2. $u = x^2 + t^2$ 3. $u = \sin 9t \sin \frac{1}{4}x$ 4. $u = \cos 4t \sin 2x$ 5. $u = \sin ct \sin x$

Heat Equation (2) (with suitable c)

6. $u = e^{-t} \sin x$ 7. $u = e^{-4t} \cos 3x$ 8. $u = e^{-9t} \cos \omega x$ 9. $u = e^{-\omega^2 c^2 t} \sin \omega x$

Laplace Equation (3)

10. $u = 2xy$ 11. $u = e^x \sin y$ 12. $u = \cos x \sinh y$ 13. $u = \arctan (y/x)$

14. **TEAM PROJECT. Verification of Solutions (a) Poisson Equation.** Verify that u satisfies (4) with $f(x, y)$ as indicated.

$$u = x^2 + y^2, \qquad f = 4$$
$$u = \cos (xy), \qquad f = -(x^2 + y^2) \cos (xy)$$
$$u = y/x, \qquad f = 2y/x^3$$

(b) **Laplace Equation.** Verify that $u = 1/\sqrt{x^2 + y^2 + z^2}$ satisfies (6).

(c) Verify that u with arbitrary (sufficiently often differentiable) v and w satisfies the given equation.

$$u = v(x) + w(y), \qquad u_{xy} = 0$$
$$u = v(x)w(y), \qquad uu_{xy} = u_x u_y$$
$$u = v(x + 2t) + w(x - 2t), \qquad u_{tt} = 4u_{xx}$$

Partial Differential Equations Solvable as Ordinary Differential Equations

If an equation involves derivatives with respect to one variable only, we can solve it like an ordinary differential equation, treating the other variable (or variables) as parameters. Find solutions $u(x, y)$ of

15. $u_y = u$ 16. $u_{xx} + 9u = 0$ 17. $u_{yy} = 0$ 18. $u_y + 2yu = 0$
19. $u_{xy} = u_x$ 20. $u_{yy} = u$ 21. $u_y = 2xyu$ 22. $u_{yy} = u_y$

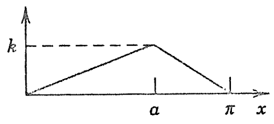
23. (Boundary value problem) Verify that $u(x, y) = a \ln (x^2 + y^2) + b$ satisfies Laplace's equation (3) and determine a and b so that u satisfies the boundary conditions $u = 0$ on the circle $x^2 + y^2 = 1$ and $u = 3$ on the circle $x^2 + y^2 = 4$. Sketch a figure of the surface represented by this function.
24. (Surface of revolution) Show that the solutions $z = z(x, y)$ of $yz_x = xz_y$ represent surfaces of revolution. Give examples. *Hint.* Use polar coordinates r, θ and show that the equation becomes $z_\theta = 0$.
25. (System) Solve the system $u_{xx} = 0$, $u_{yy} = 0$.

1. **(Frequency)** How does the frequency of the fundamental mode of the vibrating string depend on the length of the string? On the mass per unit length? On the tension? What happens to that frequency if we double the tension?

Deflection $u(x, t)$ of the String

Find $u(x, t)$ of the string of length $L = \pi$ when $c^2 = 1$, the initial velocity is zero, and the initial deflection is

2. $0.01 \sin 3x$
3. $k(\sin x - \frac{1}{2} \sin 2x)$
4. $0.1x(\pi - x)$
5. $0.1x(\pi^2 - x^2)$
6.



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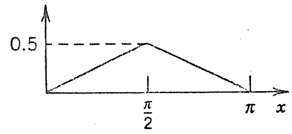

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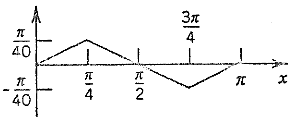

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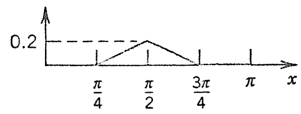

10. **(Nonzero initial velocity)** Find the deflection $u(x, t)$ of the string of length $L = \pi$ and $c^2 = 1$ for zero initial displacement and "triangular" initial velocity $u_t(x, 0) = 0.01x$ if $0 \leqq x \leqq \frac{1}{2}\pi$, $u_t(x, 0) = 0.01(\pi - x)$ if $\frac{1}{2}\pi \leqq x \leqq \pi$. (Initial conditions with $u_t(x, 0) \neq 0$ are hard to realize experimentally.)
11. **CAS PROJECT. Plots of Normal Modes.** Write a program for plotting a figure of u_n with $L = \pi$ similar to Fig. 260. Apply the program to u_2, u_3, u_4. Plot these functions as surfaces over the xt-plane. Explain the connection between these two kinds of plot.

Separation of Variables

Find solutions $u(x, y)$ of the following equations by separating variables.

12. $u_x + u_y = 0$
13. $u_x - u_y = 0$
14. $y^2 u_x - x^2 u_y = 0$
15. $u_x + u_y = (x + y)u$
16. $u_{xx} + u_{yy} = 0$
17. $u_{xy} - u = 0$
18. $u_{xx} - u_{yy} = 0$
19. $xu_{xy} + 2yu = 0$

20. **TEAM PROJECT. Forced Vibrations of an Elastic String.** Show the following.

(a) Substitution of

$$u(x, t) = \sum_{n=1}^{\infty} G_n(t) \sin \frac{n\pi x}{L} \qquad (L = \text{length of the string}) \tag{18}$$

into the wave equation (1) governing free vibrations leads to

$$\ddot{G}_n + \lambda_n^2 G = 0, \qquad \lambda_n = \frac{cn\pi}{L} \qquad \text{[see (11*)].}$$

(b) Forced vibrations of the string under an external force $P(x, t)$ per unit length acting normal to the string are governed by the equation

$$u_{tt} = c^2 u_{xx} + \frac{P}{\rho}. \tag{19}$$

(c) For a sinusoidal force $P = A\rho \sin \omega t$ we obtain

$$\frac{P}{\rho} = A \sin \omega t = \sum_{n=1}^{\infty} k_n(t) \sin \frac{n\pi x}{L}, \qquad k_n(t) = \begin{cases} (4A/n\pi) \sin \omega t & (n \text{ odd}) \\ 0 & (n \text{ even}) \end{cases} \tag{20}$$

Substituting (18) and (20) into (19) gives

$$\ddot{G}_n + \lambda_n^2 G_n = \frac{2A}{n\pi}(1 - \cos n\pi) \sin \omega t.$$

If $\lambda_n{}^2 \neq \omega^2$, the solution is

$$G_n(t) = B_n \cos \lambda_n t + B_n{}^* \sin \lambda_n t + \frac{2A(1 - \cos n\pi)}{n\pi(\lambda_n{}^2 - \omega^2)} \sin \omega t.$$

Determine B_n and $B_n{}^*$ so that u satisfies the initial conditions $u(x, 0) = f(x)$, $u_t(x, 0) = 0$.

(d) **(Resonance)** Show that if $\lambda_n = \omega$, then

$$G_n(t) = B_n \cos \omega t + B_n{}^* \sin \omega t - \frac{A}{n\pi\omega}(1 - \cos n\pi)t \cos \omega t.$$

(e) **(Reduction of boundary conditions)** Show that a problem (1)–(4) with more complicated boundary conditions $u(0, t) = 0$, $u(L, t) = h(t)$, can be reduced to a problem for a new function v satisfying conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f_1(x)$, $v_t(x, 0) = g_1(x)$ but a nonhomogeneous wave equation. *Hint.* Set $u = v + w$ and determine w suitably.

11.5

1. **WRITING PROJECT. Wave and Heat Equations.** Write a short essay on the different general behavior of the solutions of these two equations. Begin with a comparison of Figs. 263 and 267. Then explain what you can say in more general situations.
2. **(Decay)** If the first eigenfunction (9) of the bar decreases to half its value within 20 seconds, what is the value of the diffusivity c^2? How does the rate of decay of (9) for fixed n depend on the specific heat, the density, and the thermal conductivity of the material?

Bar Insulated as in the Text

Find the temperature $u(x, t)$ in a bar of silver (length 10 cm, constant cross section of area 1 cm^2, density 10.6 gm/cm^3, thermal conductivity 1.04 cal/(cm sec °C), specific heat 0.056 cal/(gm °C)) that is perfectly insulated laterally, whose ends are kept at temperature 0°C and whose initial temperature (in °C) is $f(x)$, where

3. $f(x) = \sin 0.1\pi x$ 4. $f(x) = k \sin 0.2\pi x$ 5. $f(x) = x(10 - x)$ 6. $f(x) = 2 - 0.4|x - 5|$

7. **(Different temperatures at the ends)** What is the limit $u_I(x)$ of the temperature of the bar in the text as $t \to \infty$ if the ends are kept at $u(0, t) = U_1 = const$ and $u(L, t) = U_2 = const$?
8. **(Different temperatures)** What is the temperature in the bar in Prob. 7 at any time?

Adiabatic and Other Conditions

9. **(Insulated ends, adiabatic boundary conditions)** The heat flux through the faces at the ends of a bar is found to be proportional to $u_n = \partial u/\partial n$ at the ends. Show that if the bar is perfectly insulated, also at the ends $x = 0$, $x = L$ ("adiabatic conditions"), and the initial temperature is $f(x)$, then

$$u_x(0, t) = 0, \qquad u_x(L, t) = 0, \qquad u(x, 0) = f(x)$$

and separating variables gives the solution

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{L} \exp\left[-\left(\frac{cn\pi}{L}\right)^2 t\right]$$

with $A_n = a_n$ given by (4), Sec. 10.4. Note that $u \to A_0$ as $t \to \infty$. Does this agree with your physical intuition?

Adiabatic Conditions. Find the temperature in the bar in Prob. 9 when $L = \pi$, $c = 1$, and

10. $f(x) = x$ 11. $f(x) = k = const$ 12. $f(x) = \cos 2x$ 13. $f(x) = 1 - x/\pi$

14. **(Nonhomogeneous heat equation)** Show that the problem consisting of

$$u_t - c^2 u_{xx} = Ne^{-\alpha x}$$

and (2), (3) can be reduced to a problem for the homogeneous heat equation by setting $u(x, t) = v(x, t) + w(x)$ and determining w so that v satisfies the homogeneous equation and the conditions $v(0, t) = v(L, t) = 0$, $v(x, 0) = f(x) - w(x)$. (The term $Ne^{-\alpha x}$ may represent heat loss due to radioactive decay in the bar.)

15. **The boundary condition of heat transfer**

$$-u_x(\pi, t) = k[u(\pi, t) - u_0] \qquad (21)$$

applies when a bar of length π with $c = 1$ is laterally insulated, the left end $x = 0$ is kept at 0°C, and at the right end heat is flowing into air of constant temperature u_0. Let $k = 1$ for simplicity, and $u_0 = 0$. Show that a solution is $u(x, t) = \sin px\, e^{-p^2 t}$, where p is a solution of $\tan p\pi = -p$. Show graphically that this equation has infinitely many positive solutions $p_1, p_2, p_3, \cdots$, where $p_n > n - \frac{1}{2}$ and $\lim_{n\to\infty} (p_n - n + \frac{1}{2}) = 0$. (Formula (21) is also known as radiation boundary condition, but this is misleading; see Ref. [C1], p. 19.)

16. The **heat** of a solution $u(x, t)$ across $x = 0$ is defined by $\phi(t) = -Ku_x(0, t)$. Find the heat flux for the solution (10) and note that it goes to zero as $t \to \infty$. Is this physically understandable? Explain the name.