PROBLEM SET 5.1

Laplace Transforms. Find the Laplace transforms of the following functions. Show the details of your work. (a, b, c, ω, δ are constant.)

1. $2t + 6$ **2.** $a + bt + ct^2$ **3.** $\sin \pi t$ **4.** $\cos^2 \omega t$

5. e^{a-bt} **6.** $e^t \cosh 3t$ **7.** $\sin(\omega t + \delta)$ **8.** $\sin 2t \cos 2t$

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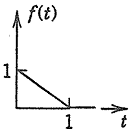

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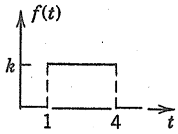

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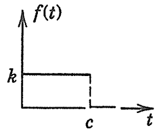

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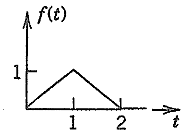

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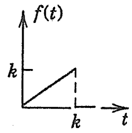

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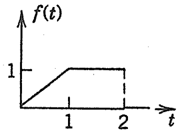

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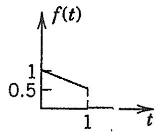

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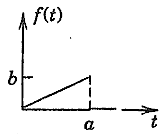


Inverse Laplace Transforms. Given $F(s) = \mathcal{L}(f)$, find $f(t)$. Show the details. (L, n, etc. are constant.)

17. $\dfrac{0.1s + 0.9}{s^2 + 3.24}$ **18.** $\dfrac{5s}{s^2 - 25}$ **19.** $\dfrac{-s - 10}{s^2 - s - 2}$

20. $\dfrac{s - 4}{s^2 - 4}$ **21.** $\dfrac{2.4}{s^4} - \dfrac{228}{s^6}$ **22.** $\dfrac{60 + 6s^2 + s^4}{s^7}$

23. $\dfrac{s}{L^2s^2 + n^2\pi^2}$ **24.** $\dfrac{1 - 7s}{(s - 3)(s - 1)(s + 2)}$ **25.** $\displaystyle\sum_{k=1}^{5} \frac{a_k}{s + k^2}$

26. $\dfrac{s^4 + 6s - 18}{s^5 - 3s^4}$ **27.** $\dfrac{1}{(s + \sqrt{2})(s - \sqrt{3})}$ **28.** $\dfrac{2s^3}{s^4 - 1}$

Applications of the First Shifting Theorem

Find the Laplace transform. (Show the details.)

29. t^2e^{-3t} **30.** $e^{-\alpha t} \cos \beta t$ **31.** $5e^{2t} \sinh 2t$

32. $2e^{-t} \cos^2 \frac{1}{2}t$ **33.** $\sinh t \cos t$ **34.** $(t + 1)^2e^t$

Find the inverse transform. (Show the details.)

35. $\dfrac{1}{(s + 1)^2}$ **36.** $\dfrac{12}{(s - 3)^4}$ **37.** $\dfrac{3}{s^2 + 6s + 18}$

38. $\dfrac{4}{s^2 - 2s - 3}$ **39.** $\dfrac{s}{(s + \frac{1}{2})^2 + 1}$ **40.** $\dfrac{2}{s^2 + s + \frac{1}{2}}$

41. **(Growth)** Prove (3).

42. **(Inverse transform)** Prove that $\mathcal{L}^{-1}$ is linear. *Hint.* Use the fact that $\mathcal{L}$ is linear.

43. **(Inverse transform)** Rewrite Table 5.1, using $\mathcal{L}^{-1}$ (e.g., $\mathcal{L}^{-1}(1/s^3) = t^2/2$).

44. **(Replacement of t by ct)** If $\mathcal{L}(f(t)) = F(t)$ and c is any positive constant, show that $\mathcal{L}(f(ct)) = F(s/c)/c$. [*Hint.* Use (1).] Use this to obtain $\mathcal{L}(\cos \omega t)$ from $\mathcal{L}(\cos t)$.

45. **(Nonexistence)** Give simple examples of functions that have no Laplace transform. Indicate the reason.

Initial Value Problems. Solve the following initial value problems by the Laplace transform. (Show the details of your work.)

1. $y' + 3y = 10 \sin t, \quad y(0) = 0$
2. $y' - 5y = 1.5e^{-4t}, \quad y(0) = 1$
3. $y' + 0.2y = 0.01t, \quad y(0) = -0.25$
4. $y'' - y' - 2y = 0, \quad y(0) = 8, \quad y'(0) = 7$
5. $y'' + ay' - 2a^2y = 0, \quad y(0) = 6, \quad y'(0) = 0$
6. $y'' + y = 2 \cos t, \quad y(0) = 3, \quad y'(0) = 4$
7. $y'' - 4y' + 3y = 6t - 8, \quad y(0) = 0, \quad y'(0) = 0$
8. $y'' + 0.04y = 0.02t^2, \quad y(0) = -25, \quad y'(0) = 0$
9. $y'' + 2y' - 3y = 6e^{-2t}, \quad y(0) = 2, \quad y'(0) = -14$

10. **PROJECT. Summary of Sec. 5.2.** (a) Compare the Laplace transform with the classical method of solving differential equations, explaining the advantages and illustrating the comparison with examples of your own.

   (b) Theorems 1 and 2 play a role different from that of Theorem 3. Explain the difference.

   (c) **Extension of Theorem 1.** Show that if $f(t)$ is continuous, except for an ordinary discontinuity (finite jump) at $t = a\ (> 0)$, the other conditions remaining the same as in Theorem 1, then (see Fig. 109)

$$\text{(1*)} \qquad \mathcal{L}(f') = s\mathcal{L}(f) - f(0) - [f(a + 0) - f(a - 0)]e^{-as}.$$

   (d) Using (1*), find the Laplace transform of $f(t) = t$ if $0 < t < 1$, $f(t) = 1$ if $1 < t < 2$, $f(t) = 0$ otherwise.

11. **Derivation by different methods** is possible for various formulas and is typical of Laplace transforms. Find $\mathcal{L}(\cos^2 t)$ (a) by using the result in Example 3, (b) by the method used in that example, (c) by expressing $\cos^2 t$ in terms of $\cos 2t$.

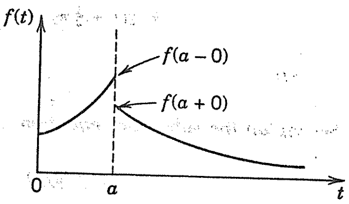


Fig. 109. Formula (1*)

12. **PROJECT. Extension of Example 4.** Extend the method of differentiation in Example 4 to obtain

(a) $\mathcal{L}(t \cos \omega t) = \dfrac{s^2 - \omega^2}{(s^2 + \omega^2)^2}$.

From this and Example 4 derive

(b) $\mathcal{L}^{-1}\left(\dfrac{1}{(s^2 + \omega^2)^2}\right) = \dfrac{1}{2\omega^3}(\sin \omega t - \omega t \cos \omega t)$

(c) $\mathcal{L}^{-1}\left(\dfrac{s}{(s^2 + \omega^2)^2}\right) = \dfrac{1}{2\omega} t \sin \omega t$

(d) $\mathcal{L}^{-1}\left(\dfrac{s^2}{(s^2 + \omega^2)^2}\right) = \dfrac{1}{2\omega}(\sin \omega t + \omega t \cos \omega t)$.

Obtain similar formulas for hyperbolic functions, namely,

(e) $\mathcal{L}(t \cosh at) = \dfrac{s^2 + a^2}{(s^2 - a^2)^2}$

(f) $\mathcal{L}(t \sinh at) = \dfrac{2as}{(s^2 - a^2)^2}$.

New Inverse Transforms by Integration (Theorem 3). Given $\mathcal{L}(f)$, find $f(t)$. (Show the details of your work.)

13. $\dfrac{1}{s^2 + 4s}$
14. $\dfrac{4}{s^3 - 2s^2}$
15. $\dfrac{1}{s(s^2 + \omega^2)}$
16. $\dfrac{1}{s^5 + s^3}$
17. $\dfrac{1}{s^3 - s}$
18. $\dfrac{1}{s^2}\left(\dfrac{s - 1}{s + 1}\right)$
19. $\dfrac{9}{s^2}\left(\dfrac{s + 1}{s^2 + 9}\right)$
20. $\dfrac{\pi^5}{s^4(s^2 + \pi^2)}$

PROBLEM SET 5.3

1. WRITING PROJECT. Shifting Theorems. Explain and compare the different roles of the two shifting theorems. Use your own formulations and examples; do not copy phrases from the text.

Applications of the Second Shifting Theorem

Laplace transform. Sketch the following functions and find their Laplace transforms. (Show the details of your work.)

2. $tu(t-1)$
3. $(t-1)u(t-1)$
4. $(t-1)^2u(t-1)$
5. $t^2u(t-1)$
6. $e^{-2t}u(t-3)$
7. $4u(t-\pi)\cos t$

Laplace Transform. Sketch the given function, which is assumed to be zero outside the given interval. Find its Laplace transform. (Show the details of your work.)

8. $t^2 \quad (0 < t < 1)$
9. $\sin \omega t \quad (0 < t < \pi/\omega)$
10. $1 - e^{-t} \quad (0 < t < 2)$
11. $e^t \quad (0 < t < 1)$
12. $\sin t \quad (2\pi < t < 4\pi)$
13. $10 \cos \pi t \quad (1 < t < 2)$

Inverse Transform. Find and sketch the inverse Laplace transform. (Show the details of your work.)

14. $4(e^{-2s} - 2e^{-5s})/s$
15. e^{-3s}/s^3
16. $e^{-3s}/(s-1)^3$
17. $3(1 - e^{-\pi s})/(s^2 + 9)$
18. $e^{-2\pi s}/(s^2 + 2s + 2)$
19. $se^{-2s}/(s^2 + \pi^2)$

Initial Value Problems. Some with Discontinuous or Impulse Inputs. Using the Laplace transform, solve the following problems. (Show the details.)

20. $4y'' - 4y' + 37y = 0, \quad y(0) = 3, \quad y'(0) = 10.5$
21. $y'' + 6y' + 8y = e^{-3t} - e^{-5t}, \quad y(0) = 0, \quad y'(0) = 0$
22. $y'' + 3y' + 2y = 4t$ if $0 < t < 1$ and 8 if $t > 1$; $y(0) = 0, \quad y'(0) = 0$
23. $y'' + 9y = 8 \sin t$ if $0 < t < \pi$ and 0 if $t > \pi$; $y(0) = 0, \quad y'(0) = 4$
24. $y'' - 5y' + 6y = 4e^t$ if $0 < t < 2$ and 0 if $t > 2$; $y(0) = 1, \quad y'(0) = -2$
25. $y'' + y' - 2y = 3 \sin t - \cos t$ if $0 < t < 2\pi$ and $3 \sin 2t - \cos 2t$ if $t > 2\pi$; $y(0) = 1, \quad y'(0) = 0$
26. $y'' + 16y = 4\delta(t - \pi), \quad y(0) = 2, \quad y'(0) = 0$
27. $y'' + y = \delta(t - \pi) - \delta(t - 2\pi), \quad y(0) = 0, \quad y'(0) = 1$
28. $y'' + 4y' + 5y = \delta(t - 1), \quad y(0) = 0, \quad y'(0) = 3$
29. $y'' + 2y' - 3y = 8e^{-t} + \delta(t - \frac{1}{2}), \quad y(0) = 3, \quad y'(0) = -5$
30. $y'' + 5y' + 6y = u(t - 1) + \delta(t - 2), \quad y(0) = 0, \quad y'(0) = 1$

Models of Electric Circuits

***RL*-Circuit.** Using the Laplace transform (and showing the details of your work), find the current $i(t)$ in the circuit in Fig. 120, assuming $i(0) = 0$ and

31. $v(t) = t$ if $0 < t < 4\pi$ and 0 if $t > 4\pi$
32. $v(t) = \sin t$ if $0 < t < 2\pi$ and 0 otherwise

***LC*-Circuit.** Using the Laplace transform (and showing the details of your work), find the current $i(t)$ in the circuit in Fig. 121, assuming $L = 1$ henry, $C = 1$ farad, zero initial current and charge on the capacitor, and

33. $v(t) = t$ if $0 < t < 1$ and $v(t) = 1$ if $t > 1$
34. $v(t) = 1$ if $0 < t < a$ and 0 otherwise
35. $v(t) = 1 - e^{-t}$ if $0 < t < \pi$ and 0 otherwise

***RC*-Circuit.** Using the Laplace transform (and showing your work), find the current $i(t)$ in the circuit in Fig. 122 with $R = 100$ ohms, $C = 0.1$ farad, and $v(t)$ volts as follows. Assume that current and charge are zero at $t = 0$.

36. $v(t) = 10000$ if $1 < t < 1.01$ and 0 otherwise
37. $v(t) = 100$ if $1 < t < 2$ and 0 otherwise. Compare with Prob. 36.
38. $v(t) = 0$ if $t < 3$ and $50(t - 3)$ if $t > 3$
39. $v(t) = 0$ if $t < 2$ and e^{-t} if $t > 2$

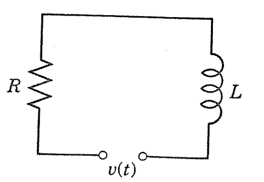


Fig. 120. Problems 31, 32

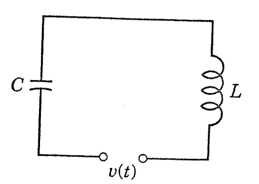


Fig. 121. Problems 33–35

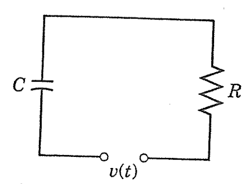


Fig. 122. Problems 36–39

PROBLEM SET 5.4

5.4

Transforms by Differentiation. Find the Laplace transform. (Show the details of your work.)

1. te^t
2. $3t \sinh 4t$
3. $t^2 \cosh \pi t$
4. $te^{-t} \cos t$
5. $t \cos \omega t$
6. $t^2 \sin 2t$
7. $te^{-t} \sin t$
8. $t^2 \cos \omega t$

Inverse Transforms by Differentiation or Integration. Using (6) or (1), find the inverse transform. (Show your work.)

9. $\dfrac{1}{(s-3)^3}$
10. $\dfrac{s}{(s^2-9)^2}$
11. $\dfrac{s^2-\pi^2}{(s^2+\pi^2)^2}$
12. $\dfrac{2s+6}{(s^2+6s+10)^2}$
13. $\ln \dfrac{s^2+1}{(s-1)^2}$
14. $\ln \dfrac{s+a}{s+b}$
15. $\dfrac{s}{(s^2+4)^2}$
16. $\text{arc cot } \dfrac{s}{\pi}$

17. **(Shifting)** Can you solve Probs. 1 and 3 by the first shifting theorem?
18. **(Differentiation)** Find $\mathcal{L}(t^n e^{at})$ by repeated application of (1), choosing $f(t) = e^{at}$.
19. **WRITING PROJECT. Differentiation and Integration of Functions and Transforms.** Make a short draft on these four operations from memory. Then compare your notes with the text and write an essay of 2–3 pages on these operations and their significance in applications.
20. **CAS PROJECT. Laguerre Polynomials.** (a) Write a CAS program for finding $l_n(t)$ in explicit form from (10). Apply it to calculate $l_0, \cdots, l_{10}$. Verify that $l_0, \cdots, l_{10}$ satisfy Laguerre's differential equation (9).

    (b) Show that

    $$l_n(t) = \sum_{m=0}^{n} \frac{(-1)^m}{m!} \binom{n}{m} t^m$$

    and calculate $l_0, \cdots, l_{10}$ from this formula.

    (c) Calculate $l_0, \cdots, l_{10}$ recursively from $l_0 = 1$, $l_1 = 1 - t$ by

    $$(n+1)l_{n+1} = (2n+1-t)l_n - nl_{n-1}.$$

CAUTION! Sometimes the functions $\tilde{l}_n = n! l_n$ are also called *Laguerre polynomials*. Their recursion is different! (Such differences in normalization are typical of several special functions—and a possible source of errors.)

Systems of Differential Equations. In Probs. 1–14 solve the given initial value problem by means of Laplace transforms. (Show the details of your work.)

1. $y_1' = -y_1 + y_2, \quad y_2' = -y_1 - y_2, \quad y_1(0) = 1, \quad y_2(0) = 0$
2. $y_1' = 6y_1 + 9y_2, \quad y_2' = y_1 + 6y_2, \quad y_1(0) = -3, \quad y_2(0) = -3$
3. $y_1' = -y_1 + 4y_2, \quad y_2' = 3y_1 - 2y_2, \quad y_1(0) = 3, \quad y_2(0) = 4$
4. $y_1' = 5y_1 + y_2, \quad y_2' = y_1 + 5y_2, \quad y_1(0) = -3, \quad y_2(0) = 7$
5. $y_1' + y_2 = 2\cos t, \quad y_1 + y_2' = 0, \quad y_1(0) = 0, \quad y_2(0) = 1$
6. $y_1'' + y_2 = -5\cos 2t, \quad y_2'' + y_1 = 5\cos 2t,$
   $y_1(0) = 1, \quad y_1'(0) = 1, \quad y_2(0) = -1, \quad y_2'(0) = 1$
7. $y_1'' = y_1 + 3y_2, \quad y_2'' = 4y_1 - 4e^t,$
   $y_1(0) = 2, \quad y_1'(0) = 3, \quad y_2(0) = 1, \quad y_2'(0) = 2$
8. $y_1'' = -5y_1 + 2y_2, \quad y_2'' = 2y_1 - 2y_2,$
   $y_1(0) = 3, \quad y_1'(0) = 0, \quad y_2(0) = 1, \quad y_2'(0) = 0$
9. $y_1' + y_2' = 2\sinh t, \quad y_2' + y_3' = e^t, \quad y_3' + y_1' = 2e^t + e^{-t},$
   $y_1(0) = 1, \quad y_2(0) = 1, \quad y_3(0) = 0$
10. $2y_1' - y_2' - y_3' = 0, \quad y_1' + y_2' = 4t + 2, \quad y_2' + y_3 = t^2 + 2,$
    $y_1(0) = y_2(0) = y_3(0) = 0$
11. $y_1' = -y_2 + 1 - u(t-1), \quad y_2' = y_1 + 1 - u(t-1), \quad y_1(0) = 0, \quad y_2(0) = 0$
12. $y_1' + y_2 = 2[1 - u(t - 2\pi)]\cos t, \quad y_1 + y_2' = 0, \quad y_1(0) = 0, \quad y_2(0) = 1$
13. $y_1' = 2y_1 - 4y_2 + u(t-1)e^t, \quad y_2' = y_1 - 3y_2 + u(t-1)e^t, \quad y_1(0) = 3, \quad y_2(0) = 0$
14. $y_1' = 2y_1 + 4y_2 + 64tu(t-1), \quad y_2' = y_1 + 2y_2, \quad y_1(0) = -4, \quad y_2(0) = -4$
15. **TEAM PROJECT. First-Order Linear Systems of Differential Equations.**
    (a) **Models.** Solve the models in Examples 1 and 2 of Sec. 3.1 by the Laplace transform method and compare the work with that in Sec. 3.1.
    (b) **Homogeneous Systems.** Solve the systems (8), (11)–(13) in Sec. 3.3 by the Laplace transform method.
    (c) **Nonhomogeneous Systems.** Solve the systems (3) and (4) in Sec. 3.6 by the Laplace transform method.
    In (a)–(c) always show the details of your work.

Further Applications

16. **(Mixing problem)** What will happen in Example 1 if you double all flows, leaving the size of the tanks and the initial conditions as before. First guess, then calculate. Can you relate the new solution to the old one?
17. **(Sinusoidal inflow from the outside)** What will happen in Example 1 if you assume that the total salt content of the inflowing 6 gal/min varies between 0 and 6 according to $6\sin^2 t$? Why is $y_2(t)$ much less wavy than $y_1(t)$?
18. **(Forced vibrations of two masses)** Solve the model in Example 3 with $k = 3$ and initial conditions $y_1(0) = 1$, $y_2(0) = 1$, $y_1'(0) = 3$, $y_2'(0) = -3$ under the assumption that the force $8\sin t$ acts on the first body and $-8\sin t$ acts on the second.
19. **(Electrical network)** Using the Laplace transform method, find the currents $i_1(t)$ and $i_2(t)$ in Fig. 132, where $v(t) = 195\sin t$ and $i_1(0) = 0$, $i_2(0) = 0$. How soon will the currents practically reach their steady state? Guess what the little curve in Fig. 132 is.
20. **(Single sine wave)** Solve Prob. 19 when the electromotive force acts from 0 to 2π only. Can you obtain the solution from that in Prob. 19 practically without calculation?

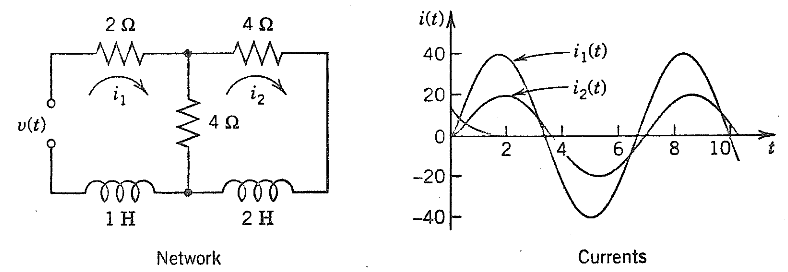


Fig. 132. Electrical network and currents in Problem 19

PROBLEM SET 10.1

10.1

Fundamental Period. Find the smallest positive period p of

1. $\cos x, \quad \sin x, \quad \cos 2x, \quad \sin 2x, \quad \cos \pi x, \quad \sin \pi x, \quad \cos 2\pi x, \quad \sin 2\pi x$
2. $\cos nx, \quad \sin nx, \quad \cos \frac{2\pi x}{k}, \quad \sin \frac{2\pi x}{k}, \quad \cos \frac{2\pi n x}{k}, \quad \sin \frac{2\pi n x}{k}$
3. **(Vector space)** If $f(x)$ and $g(x)$ have period p, show that $h = af + bg$ (a, b constant) has the period p. Thus all functions of period p form a vector space.
4. **(Integer multiples of period)** If p is a period of $f(x)$, show that np, $n = 2, 3, \cdots$, is a period of $f(x)$.
5. **(Constant)** Show that the function $f(x) = const$ is a periodic function of period p for every positive p.
6. **(Change of scale)** If $f(x)$ is a periodic function of x of period p, show that $f(ax)$, $a \neq 0$, is a periodic function of x of period p/a, and $f(x/b)$, $b \neq 0$, is a periodic function of x of period bp. Verify these results for $f(x) = \cos x$, $a = b = 2$.

Graphs of 2π-Periodic Functions

Sketch or plot the following functions $f(x)$, which are assumed to be periodic with period 2π and, for $-\pi < x < \pi$, are given by the formulas

7. $f(x) = x$
8. $f(x) = x^2$
9. $f(x) = |x|$
10. $f(x) = \pi - |x|$
11. $f(x) = |\sin x|$
12. $f(x) = e^{-|x|}$
13. $f(x) = \begin{cases} x & \text{if } -\pi \leqq x \leqq 0 \\ 0 & \text{if } 0 \leqq x \leqq \pi \end{cases}$
14. $f(x) = \begin{cases} 0 & \text{if } -\pi \leqq x \leqq 0 \\ x^2 & \text{if } 0 \leqq x \leqq \pi \end{cases}$
15. $f(x) = \begin{cases} -1 & \text{if } -\pi < x < 0 \\ 1 & \text{if } 0 < x < \pi \end{cases}$
16. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi \end{cases}$
17. $f(x) = \begin{cases} 0 & \text{if } -\pi < x < 0 \\ e^{-x} & \text{if } 0 < x < \pi \end{cases}$
18. $f(x) = \begin{cases} x^2 & \text{if } -\pi < x < 0 \\ -x^2 & \text{if } 0 < x < \pi \end{cases}$
19. **CAS PROJECT. Plotting Periodic Functions.** (a) Write a program for plotting periodic functions $f(x)$ of period 2π given for $-\pi < x \leqq \pi$. Using your program, plot the functions in Probs. 7–12 for $-10\pi \leqq x \leqq 10\pi$. Also plot some functions of your own choice.
(b) Extend your program to 2π-periodic functions given on two equal subintervals as in Probs. 13–18. Apply your program to those problems with $-10\pi \leqq x \leqq 10\pi$.
20. **CAS PROJECT. Partial Sums of Trigonometric Series.** (a) Write a program that prints a partial sum[4] of a trigonometric series (4). Applying it, list all partial sums of up to five nonzero terms of each of the series

$$\frac{1}{3}\pi^2 - 4\left(\cos x - \frac{1}{4}\cos 2x + \frac{1}{9}\cos 3x - \frac{1}{16}\cos 4x + - \cdots\right)$$

$$\frac{4}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \frac{1}{7}\sin 7x + \cdots\right)$$

$$2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + - \cdots\right).$$

(b) Plot the partial sums in (a) (for each series on common axes). Guess what periodic function the series might represent.

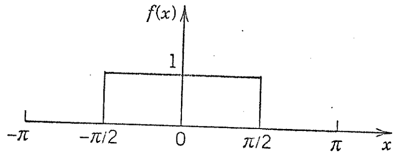
PROBLEM SET 10.2

10.2

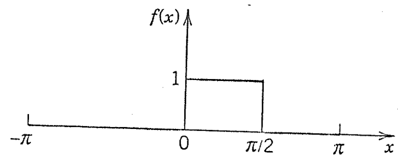
Fourier Series

Showing the details of your work, find the Fourier series of the function $f(x)$, which is assumed to have the period 2π, and plot accurate graphs of the first three partial sums, where $f(x)$ equals

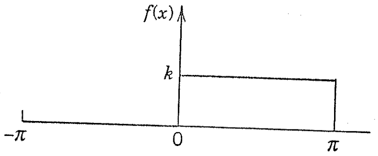
1.



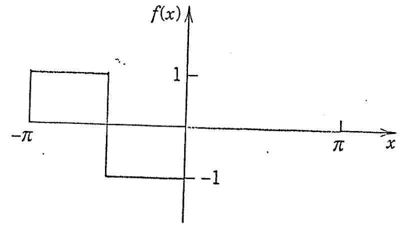
2.



3.



4.



5. $f(x) = x \quad (-\pi < x < \pi)$

6. $f(x) = x \quad (0 < x < 2\pi)$

7. $f(x) = x^2 \quad (-\pi < x < \pi)$

8. $f(x) = x^2 \quad (0 < x < 2\pi)$

9. $f(x) = x^3 \quad (-\pi < x < \pi)$

10. $f(x) = x + |x| \quad (-\pi < x < \pi)$

11. $f(x) = \begin{cases} 1 & \text{if } -\pi < x < 0 \\ -1 & \text{if } 0 < x < \pi \end{cases}$

12. $f(x) = \begin{cases} -1 & \text{if } 0 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 2\pi \end{cases}$

13. $f(x) = \begin{cases} 1 & \text{if } -\pi/2 < x < \pi/2 \\ -1 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

14. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

15. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

16. $f(x) = \begin{cases} x^2 & \text{if } -\pi/2 < x < \pi/2 \\ \pi^2/4 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$

17. **(Discontinuity)** Verify the last statement in Theorem 1 regarding discontinuities for the function in Prob. 1.

18. **CAS (Orthogonality).** Integrate and plot a typical integral, for instance, that of $\sin 3x \sin 4x$, from $-a$ to a, as a function of a, and conclude orthogonality for $a = \pi$ from the plot.

19. **CAS PROJECT. Fourier Series.** **(a)** Write a program for obtaining any partial sum of a Fourier series (7).

(b) Using the program, list all partial sums of up to five nonzero terms of the Fourier series in Probs. 5, 11, and 15, and make three corresponding plots. Comment on the accuracy.

20. **(Calculus review)** Review integration techniques for integrals as they may arise from the Euler formulas, for instance, definite integrals of $x \sin nx$, $x^2 \cos nx$, $e^{-x} \sin nx$, etc.

10.3

Fourier Series for Period $p = 2L$

Find the Fourier series of the periodic function $f(x)$, of period $p = 2L$, and sketch $f(x)$ and the first three partial sums. (Show the details of your work.)

1. $f(x) = -1 \quad (-1 < x < 0), \quad f(x) = 1 \quad (0 < x < 1), \quad p = 2L = 2$
2. $f(x) = 1 \quad (-1 < x < 0), \quad f(x) = -1 \quad (0 < x < 1), \quad p = 2L = 2$
3. $f(x) = 0 \quad (-2 < x < 0), \quad f(x) = 2 \quad (0 < x < 2), \quad p = 2L = 4$
4. $f(x) = |x| \quad (-2 < x < 2), \quad p = 2L = 4$
5. $f(x) = 2x \quad (-1 < x < 1), \quad p = 2L = 2$
6. $f(x) = 1 - x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
7. $f(x) = 3x^2 \quad (-1 < x < 1), \quad p = 2L = 2$
8. $f(x) = \frac{1}{2} + x \quad (-\frac{1}{2} < x < 0), \quad f(x) = \frac{1}{2} - x \quad (0 < x < \frac{1}{2}), \quad p = 2L = 1$
9. $f(x) = 0, \quad (-1 < x < 0), \quad f(x) = x \quad (0 < x < 1), \quad p = 2L = 2$
10. $f(x) = x \quad (0 < x < 1), \quad f(x) = 1 - x \quad (1 < x < 2), \quad p = 2L = 2$
11. $f(x) = \pi \sin \pi x \quad (0 < x < 1), \quad p = 2L = 1$
12. $f(x) = \pi x^3/2 \quad (-1 < x < 1), \quad p = 2L = 2$

13. **(Periodicity)** Show that each term in (1) has the period $p = 2L$.
14. **(Rectifier)** Find the Fourier series of the periodic function that is obtained by passing the voltage $v(t) = V_0 \cos 100\pi t$ through a half-wave rectifier.
15. **(Transformation)** Obtain the Fourier series in Prob. 1 from that in Example 1, Sec. 10.2.
16. **(Transformation)** Obtain the Fourier series in Prob. 7 from that in Prob. 7, Sec. 10.2.
17. **(Transformation)** Obtain the Fourier series in Prob. 3 from that in Example 1, Sec. 10.2.
18. **(Interval of Integration)** Show that in (2) the interval of integration may be replaced by any other interval of length $p = 2L$.
19. **CAS PROJECT. Fourier Series of $2L$-Periodic Functions.** **(a)** Write a program for obtaining any partial sum of a Fourier series (1).
    (b) Apply the program to Probs. 5–7, plotting the first few partial sums of each of the three series on common axes. Choose the first five or more partial sums until they approximate the given function reasonably well.
20. **CAS PROJECT. Gibbs Phenomenon.** The partial sums $s_n(x)$ of a Fourier series show oscillations near a discontinuity point. These do not disappear as n increases but instead become sharp "spikes." They were explained mathematically by J. W. Gibbs.[10] Plot $s_n(x)$ in Prob. 5. When $n = 20$, say, you will see those oscillations quite distinctly. Consider two other Fourier series of your choice in a similar way.

10.4

Even and Odd Functions

Are the following functions odd, even, or neither odd nor even?

1. $|x^3|$, $x \cos nx$, $x^2 \cos nx$, $\cosh x$, $\sinh x$, $\sin x + \cos x$, $x|x|$
2. $x + x^2$, $|x|$, e^x, e^{x^2}, $\sin^2 x$, $x \sin x$, $\ln x$, $x \cos x$, $e^{-|x|}$

Are the following functions $f(x)$, which are assumed to be periodic, of period 2π, even, odd or neither even nor odd?

3. $f(x) = x^2 \quad (0 < x < 2\pi)$
4. $f(x) = x^4 \quad (0 < x < 2\pi)$
5. $f(x) = e^{-|x|} \quad (-\pi < x < \pi)$
6. $f(x) = |\sin 5x| \quad (-\pi < x < \pi)$
7. $f(x) = \begin{cases} 0 & \text{if } 2 < x < 2\pi - 2 \\ x & \text{if } -2 < x < 2 \end{cases}$
8. $f(x) = \begin{cases} \cos^2 x & \text{if } -\pi < x < 0 \\ \sin^2 x & \text{if } 0 < x < \pi \end{cases}$
9. $f(x) = x^3 \quad (-\pi/2 < x < 3\pi/2)$

10. **PROJECT. Even and Odd Functions.** (a) Are the following expressions even or odd? Sums and products of even functions and of odd functions. Products of even times odd functions. Absolute values of odd functions. $f(x) + f(-x)$ and $f(x) - f(-x)$ for arbitrary $f(x)$.
    (b) Write e^{kx}, $1/(1 - x)$, $\sin(x + k)$, $\cosh(x + k)$ as sums of an even and an odd function.
    (c) Find all functions that are both even and odd.
    (d) Is $\cos^3 x$ even or odd? $\sin^3 x$? Find the Fourier series of these two functions. Do you recognize familiar identities?

Fourier Series of Even and Odd Functions

State whether the given function is even or odd. Find its Fourier series. Sketch the function and some partial sums. (Show the details of your work.)

11. $f(x) = \begin{cases} k & \text{if } -\pi/2 < x < \pi/2 \\ 0 & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
12. $f(x) = \begin{cases} -2x & \text{if } -\pi < x < 0 \\ 2x & \text{if } 0 < x < \pi \end{cases}$
13. $f(x) = \begin{cases} x & \text{if } -\pi/2 < x < \pi/2 \\ \pi - x & \text{if } \pi/2 < x < 3\pi/2 \end{cases}$
14. $f(x) = \begin{cases} x & \text{if } 0 < x < \pi \\ \pi - x & \text{if } \pi < x < 2\pi \end{cases}$
15. $f(x) = x^2/2 \quad (-\pi < x < \pi)$
16. $f(x) = 3x(\pi^2 - x^2) \quad (-\pi < x < \pi)$

Show that

17. $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + - \cdots = \frac{\pi}{4}$ (Use Prob. 11.)
18. $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{6}$ (Use Prob. 15.)
19. $1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + - \cdots = \frac{\pi^2}{12}$ (Use Prob. 15.)

Half-Range Expansions

Find the Fourier cosine series as well as the Fourier sine series. Sketch $f(x)$ and its two periodic extensions. (Show the details.)

20. $f(x) = 1 \quad (0 < x < L)$
21. $f(x) = x \quad (0 < x < L)$
22. $f(x) = x^2 \quad (0 < x < L)$
23. $f(x) = \pi - x \quad (0 < x < \pi)$
24. $f(x) = x^3 \quad (0 < x < L)$
25. $f(x) = e^x \quad (0 < x < L)$

Problem Set 10.5

10.5

1. (Calculus review) Review complex numbers.

Complex Fourier Series. Find the complex Fourier series of the following functions. (Show the details of your work.)

2. $f(x) = -1$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
3. $f(x) = x \quad (-\pi < x < \pi)$
4. $f(x) = 0$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
5. $f(x) = x \quad (0 < x < 2\pi)$
6. $f(x) = x^2 \quad (-\pi < x < \pi)$

7. **(Even and odd functions)** Show that the complex Fourier coefficients of an even function are real and those of an odd function are pure imaginary.
8. **(Conversion)** Convert the Fourier series in Prob. 5 to real form.
9. **(Fourier coefficients)** Show that $a_0 = c_0$, $a_n = c_n + c_{-n}$, $b_n = i(c_n - c_{-n})$, $n = 1, 2, \cdots$.
10. **PROJECT. Complex Fourier Coefficients.** It is very interesting that the c_n in (8) can be derived directly by a method similar to that for the a_n and b_n in Sec. 10.2. For this, multiply the series in (8) by e^{-imx} with fixed integer m and integrate termwise from $-\pi$ to π on both sides (allowed, for instance, in the case of uniform convergence), to get

$$\int_{-\pi}^{\pi} f(x)e^{-imx}\,dx = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx.$$

Show that the integral on the right equals 2π when $n = m$ and 0 when $n \neq m$ [use (5)], so that you get the coefficient formula in (8).

Problem Set 10.7

10.7

Minimum Square Error

In each case find the function $F(x)$ of the form (2) for which the total square error E on the interval $-\pi \leqq x \leqq \pi$ is minimum and compute this minimum value for $N = 1, 2, \cdots, 5$, where, for $-\pi < x < \pi$,

1. $f(x) = -1$ if $-\pi < x < 0$, $f(x) = 1$ if $0 < x < \pi$
2. $f(x) = |x|$
3. $f(x) = x$
4. $f(x) = x^2$
5. $f(x) = x^3$
6. $f(x) = x$ if $-\pi/2 < x < \pi/2$, $f(x) = \pi - x$ if $\pi/2 < x < 3\pi/2$
7. $f(x) = x$ if $-\pi/2 < x < \pi/2$, $f(x) = 0$ elsewhere in $-\pi < x < \pi$
8. $f(x) = x(\pi^2 - x^2)/12$

9. **(Monotonicity)** Show that the minimum square error (6) is a monotone decreasing function of N. How can you use this in practice? What is the smallest N in Prob. 1 for which $E^* \leqq 0.2$?
10. **CAS PROJECT. Square Error for Continuous and Discontinuous Functions.** (a) Why can you expect the decrease of the minimum square error to be more rapid for a continuous function than for a discontinuous one?
    (b) Illustrate the claim in (a) by more extensive computations for Probs. 4 and 5 and Example 1, say, for $N = 1, \cdots, 1000$.

Applications of Parseval's Identity

Using Parseval's identity, prove the following. In Probs. 11–13 compute the first few partial sums to see that the convergence is rather rapid.

11. $1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{\pi^4}{90}$ (Use Prob. 7 in Sec. 10.2.)

12. $1 + \frac{1}{9} + \frac{1}{25} + \cdots = \frac{\pi^2}{8}$ (Use Prob. 13 in Sec. 10.2.)

13. $1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \cdots = \frac{\pi^4}{96}$ (Use Prob. 13 in Sec. 10.4.)

14. $\int_{-\pi}^{\pi} \cos^4 x\,dx = \frac{3\pi}{4}$

15. $\int_{-\pi}^{\pi} \cos^6 x\,dx = \frac{5\pi}{8}$