NTNU
Institutt for matematiske fag

TMA4130 MATEMATIKK 4N
Midterm test Thursday October 27th 2005
Hours: 5:15pm – 6:45pm (90min)

Material allowed during the midterm test: Simple calculator (HP30S)
Rottman: *Booklet of formula*

N.B: Use only *one* cross for each of the exercises on the answer sheet. *Do not* write on the exercise sheet!

Exercise 1 A periodic function *f* with period 2 is defined as $f(x) = x^2$ for $-1 < x \leq 1$.

At the point $x = 99{,}8$ the Fourier series of *f* converge towards the value:

A: 0,16	**B:** −0,4	**C:** 9960,04	**D:** 0,04

We have $f(99{,}8) = f(100 - 0{,}2) = f(-0{,}2) = (-0{,}2)^2 = 0{,}04$, and the answer is **D**.

Exercise 2 The Fourier coefficient *a1* for the function defined in exercise 1 is:

A: $\frac{1}{4\pi^2}$	**B:** $-\frac{4}{\pi^2}$	**C:** 0	**D:** $\frac{2}{\pi}$

$a_1 = \int_{-1}^{1} x^2 \cos \pi x \, dx = -\frac{4}{\pi^2}$. (Use formula nr. 124, page 144 in Rottman). The answer is **B**

Exercise 3 A function *f* with the period 2 is given by $f(x) = x^9$ for $-1 < x \leq 1$. At the point *x* = -9, the Fourier series of *f* converge towards the value:

A: 1	**B:** 0	**C:** $\frac{1}{2}$	**D:** −1

Due to the periodicity, the convergence at *x* = -9 is the same as at *x* = 1. But since *f* has a discontinuity at *x* =1, with the limit value *f*(1-) = 1 and *f*(1+)= -1, we'll get 0, which mean **B.**

Exercise 4 The Laplacian transform of the function $t^2u(t-1)$ is:

A: $\frac{e^{-s}}{s^3}$ | **B:** $e^{-s}\frac{s^2+2s+2}{s^3}$ | **C:** $e^{-s}\frac{s-1}{s^3}$ | **D:** $e^{1-s}\frac{2}{s^3}$

$f(t) = (t-1+1)^2u(t-1) = [(t-1)^2+2(t-1)+1]u(t-1)$. Using the second shifting theorem, we'll get $\mathcal{L}\{f(t)\} = e^{-s}\left(\frac{2}{s^3}+\frac{2}{s^2}+\frac{1}{2}\right)$. The answer is **B.**

Exercise 5 The inverse Laplacian transform of $\frac{e^{-\pi s}}{s^2+2s+5}$ is

A: $u(t-\pi)\sin t$ | **B:** $e^{\pi-t}\cos t$

C: $(t-\pi)u(t-\pi)\sin 2t$ | **D:** $u(t-\pi)e^{\pi-t}\frac{1}{2}\sin 2t$

Factorizing the denominator gives:

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+4}\right\} = u(t-\pi)\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}(t-\pi)$$

But

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\}(t) = e^{-t}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\}(t) = e^{-t}\frac{1}{2}\sin 2t,$$

so at the end we'll get:

$$\mathcal{L}^{-1}\left\{\frac{e^{-\pi s}}{(s+1)^2+4}\right\} = u(t-\pi)e^{\pi-t}\frac{1}{2}\sin 2(t-\pi)$$

and since $sin(2t-2\pi) = sin2t$, **D** is the correct answer.

Exercise 6 The solution *y(t)* of the initial value problem

$$y'' + y = 3\cos 2t, \qquad y(0) = y'(0) = 0,$$

has the Laplacian transformed *Y(s)* given by:

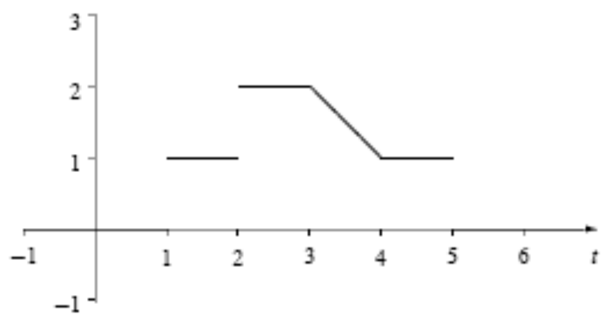
A: $\frac{3s}{(s^4+5s^2+4)}$	B: $\frac{6}{(s^2+1)(s^2+4)}$	C: $\frac{3e^{-2s}}{s^2+1}$	D: $\frac{3}{s(s^2+1)}$

The Laplacian transformation gives

$$(s^2+1)Y(s) = 3\frac{s}{s^2+4}$$

The correct answer is **A.**

Exercise 7 The function with the given graph



is given by:

A: $u(t-1)+u(t-2)-(t-3)u(t-3)-u(t-5)$

B: $u(t-1)+u(t-2)-(t-3)u(t-3)+(t-4)u(t-4)-u(t-5)$

C: $u(t-1)+u(t-2)-(t-3)u(t-3)+tu(t-4)-u(t-5)$

D: $u(t-1)+u(t-2)-tu(t-3)+tu(t-4)-u(t-5)$

The correct answer is **B**.

Exercise 8

The convolution product (refer to page 176 in Rottmann, the booklet of formulas,, for definition of the convolution product) 1 * *cos t* is equal to

A: $t\cos t$	B: $\cos t$	C: $\sin t$	D: te^{-t}

The answer is **C.** Either by calculation using the definition or using the Fourier transformation:

$$\mathcal{L}\{1 * \cos t\} = \frac{1}{s} \cdot \frac{s}{s^2+1} = \frac{1}{s^2+1} = \mathcal{L}\{\sin t\}$$

(And then take the inverse Laplacian transformation).

Exercise 9

Let the function *f(x)* be given by

$$f(x) = \begin{cases} 1 & \text{for } |x| \le 1 \\ 0 & \text{ellers.} \end{cases}$$

The Fourier transformed $\hat{f}(w)$ is given by:

A: $\dfrac{\pi w}{1+w^2}$ **B:** $\dfrac{1}{\sqrt{2\pi}} \dfrac{e^{iw}-1}{w}$ **C:** $\sqrt{\dfrac{2}{\pi}} \dfrac{\sin w}{w}$ **D:** $\sqrt{\dfrac{2}{\pi}} \dfrac{\cos w}{w}$

We then have

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{-iwx}\, dx = \frac{1}{\sqrt{2\pi}} \frac{e^{iw} - e^{-iw}}{iw} = \frac{2}{\sqrt{2\pi}} \frac{\sin w}{w}.$$

The answer is therefore **C**.

Exercise 10

The value of the integration

$$\int_0^\infty \arctan \frac{2}{w^2}\, dw$$

is (*Hint:* Use that the Fourier transformation of $f(x) = \dfrac{e^{-|x|}\sin x}{x}$ is $\hat{f}(w) = \dfrac{1}{\sqrt{2\pi}} \arctan \dfrac{2}{w^2}$.
)

A: $\frac{\pi}{2}$ **B:** 2π **C:** π **D:** $\frac{\pi^3}{32}$

We use the hint provided. Notice that if we define $f(0) = 1$, then f is continuous all over, since $lim_{x\to 0}\frac{sin x}{x} = 1$. We then have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx}\, dw$$

and by setting in $x = 0$ yields

$$1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \arctan \frac{2}{w^2}\, dw = \frac{1}{\pi} \int_{0}^{\infty} \arctan \frac{2}{w^2}\, dw$$

where the last equality follows from the fact that $arctan\frac{2}{w^2}$ is an even function of w. The answer is therefore **C**.