NTNU
Institutt for matematiske fag

Finite Dimensional Normed Spaces

Theorem. *If E is a finite-dimensional vector space and $\|\cdot\|_1$ and $\|\cdot\|_2$ are two norms on E, then they are equivalent norms, i.e. there exists $\alpha, \beta > 0$ such that*

$$\alpha\|x\|_1 \leq \|x\|_2 \leq \beta\|x\|_1$$

for all $x \in E$.

In order to prove this we need a lemma that we do not prove.

Lemma. *If $A \subseteq \mathbb{R}^n$ is closed and bounded, and $f : A \to \mathbb{R}$ is continuous, then there are points $t_0, t_1 \in A$ such that $f(t_0) \leq f(t) \leq f(t_1)$ for all $t \in A$.* $\square$

Proof of the theorem. Let $\|\cdot\|$ be a norm on E. Fix a basis $e_1, \ldots, e_n$ of E and define $f : \mathbb{C}^n \to \mathbb{R}$ by

$$f(t) = \|t_1 e_1 + \cdots + t_n e_n\|.$$

Then we have that

$$f(t) \leq \sum_{j=1}^{n} |t_j|\|e_j\| \leq \Big(\sum_{j=1}^{n} |t_j|^2\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{n} \|e_j\|^2\Big)^{\frac{1}{2}},$$

and hence

$$f(t) \leq b\|t\|$$

for all $t \in \mathbb{C}^n$. Here $b = \big(\sum_{j=1}^{n} \|e_j\|^2\big)^{\frac{1}{2}} > 0$ and $\|t\|$ also denotes the norm in $\mathbb{C}^n$.

Let $A = \{t \mid \|t\| = 1\} \subseteq \mathbb{C}^n$. Then A is both closed and bounded in $\mathbb{C}^n$, and $f : A \to \mathbb{R}$ is continuous. By the lemma there is a $t_0 \in A$ such that $f(t_0) \leq f(t)$ for all $t \in A$. Let $a = f(t_0) > 0$. For any $t \neq 0$ in $\mathbb{C}^n$ we have that $\frac{1}{\|t\|}t \in A$, and $a \leq f\big(\frac{1}{\|t\|}t\big) = \frac{f(t)}{\|t\|}$. Thus we have

$$a\|t\| \leq f(t)$$

for all $t \in \mathbb{C}^n$.

Hence for $x = \sum_{j=1}^{n} t_j e_j$ we have

$$a\|t\| \leq \|x\| \leq b\|t\|$$

where $a, b > 0$. For the two given norms we then have

$$a_1\|t\| \leq \|x\|_1 \leq b_1\|t\|$$
$$a_2\|t\| \leq \|x\|_2 \leq b_2\|t\|.$$

Thus

$$\frac{a_2}{b_1}\|x\|_1 \leq a_2\|t\| \leq \|x\|_2 \leq b_2\|t\| \leq \frac{b_2}{a_1}\|x\|_1,$$

and we may take $\alpha = \frac{a_2}{b_1}$ and $\beta = \frac{b_2}{a_1}$. $\square$

Corollary. *Any finite dimensional normed space is a Banach space. In particular $\mathbb{C}^n$ and $\mathbb{R}^n$ are Banach spaces with respect to any norm.*

Proof. If $x = \sum_{j=1}^n t_j e_j$ as in the proof above, then we have ($\|\cdot\|$ denotes the norm on both E and $\mathbb{C}^n$)

$$a\|t\| \overset{(1)}{\leq} \|x\| \overset{(2)}{\leq} b\|t\|.$$

Let $(x^{(m)})_{m=1}^\infty$ be a Cauchy sequence in E, and write $x^{(m)} = \sum_{j=1}^n t_j^{(m)} e_j$ where $t^{(m)} = \big(t_1^{(m)}, \dots, t_n^{(m)}\big) \in \mathbb{C}^n$. Then (1) shows that $(t^{(m)})_{m=1}^\infty$ is a Cauchy sequence in $\mathbb{C}^n$. $\mathbb{C}^n$ is complete, so let $t = \lim_{m\to\infty} t^{(m)} \in \mathbb{C}^n$ and $x = \sum_{j=1}^n t_j e_j \in E$. Using (2) we have

$$\|x^{(m)} - x\| \leq b\|t^{(m)} - t\|$$

and this gives that $\lim_{m\to\infty} x^{(m)} = x$ in E. Hence E is complete. The rest is then clear. □

Recall from Problem Set No. 2, Problem 3 b) that

$$\|x\|_\infty \leq \|x\| \leq \sqrt{n}\|x\|_\infty$$

and

$$\|x\|_\infty \leq \|x\|_1 \leq n\|x\|_\infty$$

in $\mathbb{R}^n$ (or $\mathbb{C}^n$) where

$$\|x\|_\infty = \max_{1\leq j\leq n} |x_j|$$

$$\|x\|_1 = \sum_{j=1}^n |x_j|$$

$$\|x\| = \Big(\sum_{j=1}^n |x_j|^2\Big)^{\frac{1}{2}}.$$

(IH; October 24, 2006)