Norwegian University of Science and Technology
Department of Mathematical Sciences

Contact during the exam: Idar Hansen
(Phone: 73593528)

EXAM IN TMA4145 LINEAR METHODS

Monday December 18, 2006
Time: kl. 15.00 - 19.00
Permitted aids (Code D): Approved calculator (HP30S),
No handwritten or printed material allowed.
English

Grades: January 18, 2007

Problem 1

a) State Banach's Fixed Point Theorem.

b) Show that if $X \neq \emptyset$ is a complete metric space, and $f : X \to X$ is a function such that $f^2 = f \circ f$ is a contraction, then f has exactly one fixed point.

c) Consider $C[0,1]$ with the metric d_∞, and let $F : C[0,1] \to C[0,1]$ be given by

$$(Fx)(t) = t + \int_0^t x(s)ds,\ 0 \leq t \leq 1.$$

Show that F has a unique fixed point x^*, and use iteration to find x^*.

Problem 2

Let

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find a singular value decomposition of A, and use the pseudo-inverse A^+ of A to solve the least squares problem

$$Ax = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}.$$

Problem 3

Given

$$A = \begin{bmatrix} 3 & -1 & -1 \\ 0 & 2 & 0 \\ 1 & -1 & 1 \end{bmatrix}.$$

a) Find a Jordan form J of A.

Find a matrix S such that $S^{-1}AS = J$.

b) Solve the differential equation $u' = Au$.

Problem 4

a) Let $(e_n)_{n=1}^{\infty}$ be an orthonormal sequence in a Hilbert space H, and let $(\lambda_n)_{n=1}^{\infty}$ be a sequence of complex numbers.

Show that $\Sigma_{n=1}^{\infty} \lambda_n e_n$ is convergent in H if and only if $\Sigma_{n=1}^{\infty} |\lambda_n|^2 < \infty$.

b) Find $a, b \in \mathbb{C}$ such that

$$\int_0^1 |e^t - a - bt|^2 dt$$

is minimal.

Problem 5

Consider the subspace

$$M = \{x | x(t) = 0 \quad \text{for} \quad 0 \le t \le \tfrac{1}{2}\}$$

of $C[0,1]$, and let $C[0,1]$ have the norm derived from the inner product given by

$$\langle x, y \rangle = \int_0^1 x(t)\overline{y(t)}dt.$$

a) Show that if $x \in C[0,1]$ and $y \in M$, then

$$\int_0^{\frac{1}{2}} |x(t)|^2 dt \le \|x - y\|^2.$$

Show that M is a closed subset of $C[0,1]$.

b) Show that $\|x - 1\| \ge \frac{1}{\sqrt{2}}$ for all $x \in M$. Does there exist an element $x_0 \in M$ such that $\|x_0 - 1\| = \frac{1}{\sqrt{2}}$? (Here 1 denotes the constant function $1(t) = 1$ for $0 \le t \le 1$.)