

Opp 2

a) First we need to show that G is closed under binary operation:

$$\begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \begin{pmatrix} a' & b' & c' \\ 0 & a' & b' \\ 0 & 0 & a' \end{pmatrix} = \begin{pmatrix} aa' & ab'+ba' & ac'+bb'+ca' \\ 0 & aa' & ab'+ba' \\ 0 & 0 & aa' \end{pmatrix}$$

We see that the product is in G since $aa' \neq 0$ when $a \neq 0 \neq a'$.

Since multiplication of matrices is associative, we do not need to check that.

The identity $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is in G .

$$\begin{pmatrix} a & b & c & | & 1 & 0 & 0 \\ 0 & a & b & | & 0 & 1 & 0 \\ 0 & 0 & a & | & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & \frac{b}{a} & \frac{c}{a} & | & \frac{1}{a} & 0 & 0 \\ 0 & 1 & \frac{b}{a} & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & \frac{1}{a} \end{pmatrix} \\ \sim \begin{pmatrix} 1 & \frac{b}{a} & 0 & | & \frac{1}{a} & 0 & -\frac{b}{a^2} \\ 0 & 1 & 0 & | & 0 & 1 & -\frac{b}{a^2} \\ 0 & 0 & 1 & | & 0 & 0 & \frac{1}{a} \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & | & \frac{1}{a} & -\frac{b}{a^2} & \frac{b^2}{a^3} - \frac{c}{a^2} \\ 0 & 1 & 0 & | & 0 & 1 & -\frac{b}{a^2} \\ 0 & 0 & 1 & | & 0 & 0 & \frac{1}{a} \end{pmatrix}$$

We observe that the inverse of every element in G is still in G . Hence G is a subgroup of $GL_3(\mathbb{Z}_p)$.

b) Since b & c can be chosen freely, and $a \neq 0$, we get

$$|G| = (p-1)p^2$$

c) Look at the set $H = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid b, c \in \mathbb{Z}_p \right\}$. It is easy to check that this is a subgroup of G . Now

$$\begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & nb & nb^2 + nc \\ 0 & 1 & nb \\ 0 & 0 & 1 \end{pmatrix}$$

therefore all elements in H has order p . Since $|H| = p^2$ this shows us that H is a Sylow p -subgroup of G .

Let $r = \# \{ \text{Sylow } p\text{-subgroups} \}$. Then $r \mid |G|$ & $r \equiv 1 \pmod{p}$. Since $p \equiv 0 \pmod{p}$, the only possibilities for r are 1 & $p-1$.

$p-1 \equiv -1 \equiv 1 \pmod{p} \Rightarrow p=2 \Rightarrow r=1$ always. Hence H is the only Sylow-subgroup of G .