TMA4150/MA2201 - ANSWERS TO MIDTERM EXAM 2007

Problem 1 The first and fifth are groups. The second and fourth are not closed under multiplication mod 6 and permutation product, respectively, and the third lacks left inverses.

Problem 2 Since $100 = 2^2 \cdot 5^2$, there are four non-isomorphic groups of order 100:

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5, \quad \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_5, \quad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}, \quad \mathbb{Z}_4 \times \mathbb{Z}_{25}$$

Problem 3 The order of the element is $\operatorname{lcm}(3, 5, 5) = 15$.

Problem 4 $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10} \simeq \mathbb{Z}_3 \times (\mathbb{Z}_4 \times \mathbb{Z}_9) \times (\mathbb{Z}_2 \times \mathbb{Z}_5) \simeq \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5$

Problem 5 Note: There is a mistake in the Problem statement. There are only two homomorphisms. ϕ_1 and ϕ_5 are homomorphisms.

Problem 6 The order of σ is 6, so the index is $\frac{6!}{6} = 5! = 120$.

Problem 7

- **True :** By Lagrange's theorem, N must have order 7 or 11, which gives G/N an order of 11 or 7, respectively. In both cases, G/N has prime order, and must be cyclic.
- **False :** $H_1 \cup H_2$ is not necessarily closed under the operation. For an example, consider $H_1 = 3\mathbb{Z} \leq \mathbb{Z}$ and $H_2 = 5\mathbb{Z} \leq \mathbb{Z}$. $3 + 5 = 8 \notin 3\mathbb{Z} \cup 5\mathbb{Z}$.
- **False :** This is not true in general unless ϕ is 1-1. Consider for instance $\phi : \mathbb{Z} \to \mathbb{Z}_k$ which sends n to its residue modulo k.
- **True :** There are three subgroups of order 2 (cyclic subgroups generated by the transpositions) and one of order 3 (the alternating subgroup, consisting of the 3-cycles and the identity).

Problem 8 $G/H \simeq (\mathbb{R}^*, \cdot)$. (Consider $\phi : G \to \mathbb{R}^*$, $\phi(X) = \det(X)$. Since $\det(XY) = \det(X)\det(Y)$, it is a homomorphism. Moreover, it is onto (check this!) and the kernel is H.)

Problem 9 The second argument is correct.

Problem 10 The symmetry group is D_4, which has 8 elements.

$$\text{Number of colourings: } = \frac{1}{8}\left(4^4 + 4 + 4^2 + 4 + 4^3 + 4^3 + 4^2 + 4^2\right) = 55$$