TMA4150/MA2201 - MIDTERM EXAM 2007

Monday, March 12, 2007 - **English**

Student number: ___________

The exam consists of 10 multiple choice problems. In some of the problems, there are more than one correct answer, and the number of correct answers should be clear from the problem statement. Put **exactly** as many crosses as there are correct answers. Good luck!

Problem 1 Which **two** of the following sets are groups under the given binary operation?

- ☐ The set of 3×3 matrices with real coefficients and positive determinant under matrix multiplication
- ☐ $\{1, 2, 3, 4, 5\}$ under multiplication modulo 6
- ☐ $(\mathbb{R}, *)$, where $*$ is given by $a * b = a - b$
- ☐ The set of odd permutations in S_5 under ordinary product of permutations
- ☐ $(\mathbb{R}_{>0}, *)$, the positive real numbers, where $a * b = \frac{ab}{2}$

Problem 2 How many non-isomorphic abelian groups are there of order 100?

☐ 1 ☐ 3 ☐ 4 ☐ 9 ☐ 15

Problem 3 What is the order of the element $(6, 8, 12)$ in the group $\mathbb{Z}_9 \times \mathbb{Z}_{10} \times \mathbb{Z}_{15}$?

☐ 3 ☐ 5 ☐ 12 ☐ 15 ☐ 18

Problem 4 The group $\mathbb{Z}_3 \times \mathbb{Z}_{36} \times \mathbb{Z}_{10}$ is isomorphic to exactly **one** of the following groups. Which one?

- ☐ $\mathbb{Z}_8 \times \mathbb{Z}_9 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
- ☐ $\mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \times \mathbb{Z}_5$
- ☐ $\mathbb{Z}_3 \times \mathbb{Z}_{360}$
- ☐ $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5$
- ☐ $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_{24}$

Problem 5 Which **three** of the maps below are group homomorphisms?

- ☐ $\phi_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \quad \phi_1(a, b) = a - b$
- ☐ $\phi_2 : (\mathbb{R}, +) \to (\mathbb{R}, +), \quad \phi_2(x) = \sqrt{x}$
- ☐ $\phi_3 : S_4 \to S_5, \quad \phi_3\left(\begin{pmatrix} 1 & 2 & 3 & 4 \\ a_1 & a_2 & a_3 & a_4 \end{pmatrix}\right) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a_1 & a_2 & 3 & a_3 & a_4 \end{pmatrix}$
- ☐ $\phi_4 : S_5 \to S_5, \quad \phi_4(\sigma) = \sigma^{-1}$
- ☐ $\phi_5 : (\mathbb{R}^*, \cdot) \to (\mathbb{R}^*, \cdot), \quad \phi_5(x) = |x|$

Problem 6 Let

$$\sigma = (143)(25) \in S_6$$

How many left cosets belong to the subgroup $\langle \sigma \rangle \leq S_6$ generated by σ?

☐ 5 ☐ 6 ☐ 90 ☐ 120 ☐ 144

Problem 7 Which **two** of the following statements are correct?

- ☐ If G is a group of order 77, and N is a normal subgroup of G (where $\{e\} \neq N \neq G$), then the factor group G/N is cyclic.
- ☐ If H_1 and H_2 are subgroups of G, then also the union $H_1 \cup H_2$ is a subgroup of G.
- ☐ If $\phi : G \to G'$ is an arbitrary homomorphism, and $a \in G$ has order n, then $\phi(a) \in G'$ has order n as well.
- ☐ The number of subgroups of S_3, except from $\{e\}$ and S_3 itself, is 4.

Problem 8 Let $G = \mathrm{GL}(2, \mathbb{R})$ be the group of invertible 2×2-matrices with real coefficients under matrix multiplication. Let H be the normal subgroup

$$H = \{X \in G \mid \det(X) = 1\}$$

The factor group G/H is isomophic to exactly **one** of the groups below. Which one?

- ☐ $(\mathbb{R}_{>0}, \cdot)$, the positive real numbers under multiplication
- ☐ $(\mathbb{R}^*, \cdot)$, the real numbers $\neq 0$ under multiplication
- ☐ $(\mathbb{R}, +)$, the real numbers under addition
- ☐ $(\mathbb{C}, +)$, the complex numbers under addition
- ☐ $(\mathbb{C}^*, \cdot)$, the complex numbers $\neq 0$ under multiplication

Problem 9 Let G be a group and $H \leq G$ a subgroup. Let V_H be the set of left cosets of H. Exactly **one** of the arguments below is a correct proof for the fact that V_H is a G-set under $*$ when $g * (xH) = (gx)H$. Which one?

- ☐ If $yH = xH$, then $y = xh$ for some $h \in H$, so $g * (yH) = g * (hxH) = g * (xh'H) = g * (xH)$ (where $h' \in H$), so the action is well defined. Furthermore, $e * (xH) = (ex)H = xH$, and $g_1 * (g_2 * (xH)) = g_1 * ((g_2x)H) = (g_1g_2x)H = (g_1g_2) * (xH)$, so the requirements for a group action are fulfilled.
- ☐ If $yH = xH$, then $y = xh$ for some $h \in H$, and $g * (yH) = (gy)H = (gxh)H = (gx) * (hH) = (gx) * H = (gx)H = g * (xH)$, so the action is well defined. Also, we have that $e * (xH) = (ex)H = xH$, and $g_1 * (g_2 * (xH)) = g_1 * ((g_2x)H) = (g_1g_2x)H = (g_1g_2) * (xH)$ for all $g_1, g_2 \in G$, so the requirements for a group action are fulfilled.
- ☐ If $yH = xH$, then $y = xh$ for some $h \in H$, and $g*(yH) = g*((xh)H) = g*(xH)$, so the action is well defined. We have that $g_1 * (g_2 * (xH)) = g_1 * (g_2x)H = (g_1g_2x)H = (g_1g_2) * (xH)$, and $g*(xy)H = g*((xH)\cdot(yH)) = (gx)H \cdot (gy)H = (g*(xH))\cdot(g*(yH))$. Thus the requirements for a group action are fulfilled.

Problem 10 In how many different ways can we colour the vertices in a square, when we have four different colours to choose from and can use each colour as many times as we want? Two ways are considered to be equal if we can not distinguish them when the square is free to move in space.

☐ 12 ☐ 55 ☐ 60 ☐ 92 ☐ 256

Tip: You can use Burnside's formula, which is given by

$$r \cdot |G| = \sum_{g \in G} |X_g|$$