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Contact during exam:
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FINAL EXAM IN ALGEBRA AND NUMBER THEORY (TMA4150)
English

Saturday, May 20, 2006
Time: 09.00 – 13.00

Permitted aids:
Calculator HP30S

The exam consist of 6 problems. You should give the reasons for all your answers. Good luck!

Problem 1

a) Find all abelian groups of order 36, up to isomorphism.

b) Let G be the group of units in the ring $\mathbb{Z}_7 \times \mathbb{Z}_7$. Which of the groups in a) is G isomorphic to?

Problem 2

a) Let R be a commutative ring with multiplicative identity 1. Let U be the set of units in R. Show that U is a group under multiplication.

b) Let $R = \mathbb{Z}/n\mathbb{Z}$, $n > 1$, and explain briefly how Euler's Theorem follows from problem **a)**. (Euler's Theorem says that if a is an integer relatively prime to n, then $a^{\phi(n)} \equiv 1 (\mathrm{mod}\ n)$, where ϕ is Euler's ϕ-function.)

Problem 3 Let G be the group of invertible 2×2 matrices over the rational numbers $\mathbb{Q}$. Let $r < s$ be in $\mathbb{Q}$, $r, s \neq 0$. Let

$$H_{r,s} = \{A \in G \mid \det A = r \text{ or } \det A = s\}$$

a) Show that $H_{r,s}$ is a subgroup of G if and only if $(r, s) = (-1, 1)$.

b) Show that $H = H_{-1,1}$ is a normal subgroup of G, and that the factor group G/H is isomorphic to the group of positive rational numbers under multiplication.

Problem 4 We are going to colour the corners of a regular pentagon. Two colourings are regarded as equal if we can get one from the other by rotating and turning the pentagon in space.

a) Describe the elements of the symmetry group of the pentagon, regarded as a subgroup of the group of permutations of the five corners.

b) In how many different ways can we colour the corners of the pentagon, if we have 3 different colours to choose from, and we can use these on as many corners as we want?

Problem 5

a) If R is a commutative ring with multiplicative unity $1 \neq 0$, then so is the polynomial ring $R[x]$. (You are not supposed to show this.) Show that if R is an integral domain, then so is $R[x]$.

b) Let $p(x) = x^5 + 2x^4 + 2x^3 + 1$ be a polynomial in $\mathbb{Z}_3[x]$. Write $p(x)$ as a product of polynomials which are irreducible in $\mathbb{Z}_3[x]$.

Problem 6 Let $f(x) = x^4 + x + 1$ be a polynomial in $\mathbb{Z}_2[x]$. Explain why $F = \mathbb{Z}_2[x]/\langle f(x) \rangle$ is a field, and find a generator for the cyclic group $F \backslash \{0\}$.