EXAM
TMA4165 DIFFERENTIAL EQUATIONS AND DYNAMICAL SYSTEMS V18

General comments:

- You are expected to compare your solution with the solution posted on the webpage, before you ask for an explanation of your grade or appeal your grade.
- In case you want an explanation:
  - Send in the request the official way.
  - In case that you only have one or two specific questions about the grading of your exam, send me an email with your candidate number and I will answer them in the explanation for your grade.
  - I will not reveal your score.
- In case you appeal your grade: I cannot answer (administrative or other) questions about your appeals. You will find the correct contact information on the webpage where you handed in your appeal.

Remarks on your answers:

Problem 1:
If you have problems finding out how the phase paths look like, draw the isoclines.

Problem 2:
All of you overlooked that when one draws a phase diagram, one obtains the phase paths and the orientation, i.e. the direction in which one moves along the different phase paths, but one has no information about the speed. Thus it could be that the mirrored phase path with orientation looks the "same", but you move with a different speed, since

$$\frac{dy}{dx}\Big|_{(x,y)} = \frac{g(x,y)}{f(x,y)} = \frac{a(x,y)g(x,y)}{a(x,y)f(x,y)}.$$

Problem 3:
Bifurcation point means that one has a change in stability. In this case it is the stability of the limit cycles C_2 and C_μ.

Problem 4a:
To find and classify the EPs went quite well, however quite a few of you overlooked that you should also determine the stability of the EPs.

Therefore once more: Always read the problem you work on carefully and check afterwards that you answered all questions.

Problem 4b:

In case that the EP $(-4, 6)$ in 4a) has been identified as a stable or unstable spiral (which is wrong, it is a centre), you can still draw the phase diagram without problems (see last page).
In general: If you got a wrong answer in 4a), you still get the full score in 4b) if your phase diagram is correct with respect to your classification in 4a).

Problem 5:

The problem is based on Theorem 3 in the notes by Harald. A solution is called global if its maximal interval of existence equals $\mathbb{R}$. An example of an initial value problem where this is not the case is given by

$$\dot{x}(t) = 1 + x^2(t), \quad x(0) = 0,$$

which has as a solution $x(t) = \tan(t)$, which is only defined for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

Some of you tried to sketch the phase diagram. The phase paths are going to cover all of $\mathbb{R}^2$, but the phase paths contain no information about the speed at which you are traveling along the phase path. This means in particular, that the phase diagram gives you no information whether or not the solutions tending to "∞" are global or local.

Problem 6:

The problem is based on Theorem 8.15 in the book. Some of you checked that all elements $c_{i,j}(t)$ in the matrix $C(t)$ satisfy

$$\lim_{t\to\infty} c_{i,j}(t) = 0$$

instead of Theorem 8.15 (ii) and concluded that all solutions are asymptotically stable. Having a close look at the proof of Theorem 8.15 reveals that if one replaces (ii) by

$$\text{For } t_0 \leq t < \infty, C(t) \text{ is continuous and } \lim_{t\to\infty} \|C(t)\| = 0, \tag{0.1}$$

in Theorem 8.15, the theorem still holds. So this is fine.
Outline of the proof: Follow the proof of Theorem 8.15 until you end up with (8.46). What is left to show is that

$$\beta M e^{-mt_0} \int_{t_0}^{t} \|C(s)\| ds - mt \to -\infty \quad \text{as } t \to \infty.$$

By our new assumption, we know that there exists $T > 0$ such that

$$2\beta M e^{-mt_0} \|C(s)\| \leq m \quad \text{for all } s \geq T,$$

and we can write for all $t \geq T$,

$$\begin{aligned}
\beta M e^{-mt_0} \int_{t_0}^{t} \|C(s)\| ds - mt &= \beta M e^{-mt_0} \int_{t_0}^{T} \|C(s)\| ds - mT \\
&\quad + \beta M e^{-mt_0} \int_{T}^{t} \|C(s)\| ds - m(t-T) \\
&\leq \beta M e^{-mt_0} \int_{t_0}^{T} \|C(s)\| ds - mT \\
&\quad + \frac{m}{2}(t-T) - m(t-T) \\
&= \beta M e^{-mt_0} \int_{t_0}^{T} \|C(s)\| ds - mT - \frac{m}{2}(t-T).
\end{aligned}$$

Here the first two terms are bounded, while the third tends to $-\infty$, which proves the claim.
Warning: Theorem 8.15 and the above modification are neither equivalent, nor is one more general than the other one.
The function $f : [1, \infty) \to \mathbb{R}$, $f(t) = \frac{1}{t}$ satisifies

$$\lim_{t\to\infty} f(t) = 0 \quad \text{and} \quad \int_1^\infty f(t) = \infty.$$

There exist continuous, bounded, and piecewise linear functions $g : [1, \infty) \to \mathbb{R}$, such that

$$\int_1^\infty |g(t)| dt < \infty \qquad \text{and} \qquad \lim_{t\to\infty} g(t) \text{ does not exist.}$$

In particular, they can be chosen to satisfy a generalized version of Theorem 1 in the notes by Harald, so that one has existence and uniqueness of global solutions.

Some of you ended up with a wrong splitting and hence $\lim_{t\to\infty} c_{i,j}(t) = 3$, for example. In this case one can no longer apply Theorem 8.15 or the above modification, except for some special cases. However, the limit tells you that your matrix $C(t)$ is wrong and that you should replace $c_{i,j}(t)$ by $c_{i,j}(t) - 3$ and $a_{i,j}$ by $a_{i,j} + 3$, in our example.

In general: Try to apply the theorems, lemmas, and methods you learn in the course!

Problem 7a:

Exercise E7 is using the same idea, although the exam problem has been a bit more involved.

Problem 7b:

Some of you noticed that one can solve the given differential equation explicitly with the help of partial fraction and then use the explicit solution to compute the difference $|x^0(t) - x^\varepsilon(t)|$. This is of course ok!

Outline of this approach: The solution of (3) is given by

$$x^\varepsilon(t) = \frac{(1+\varepsilon^2)e^t}{1+2\varepsilon^2+e^t},$$

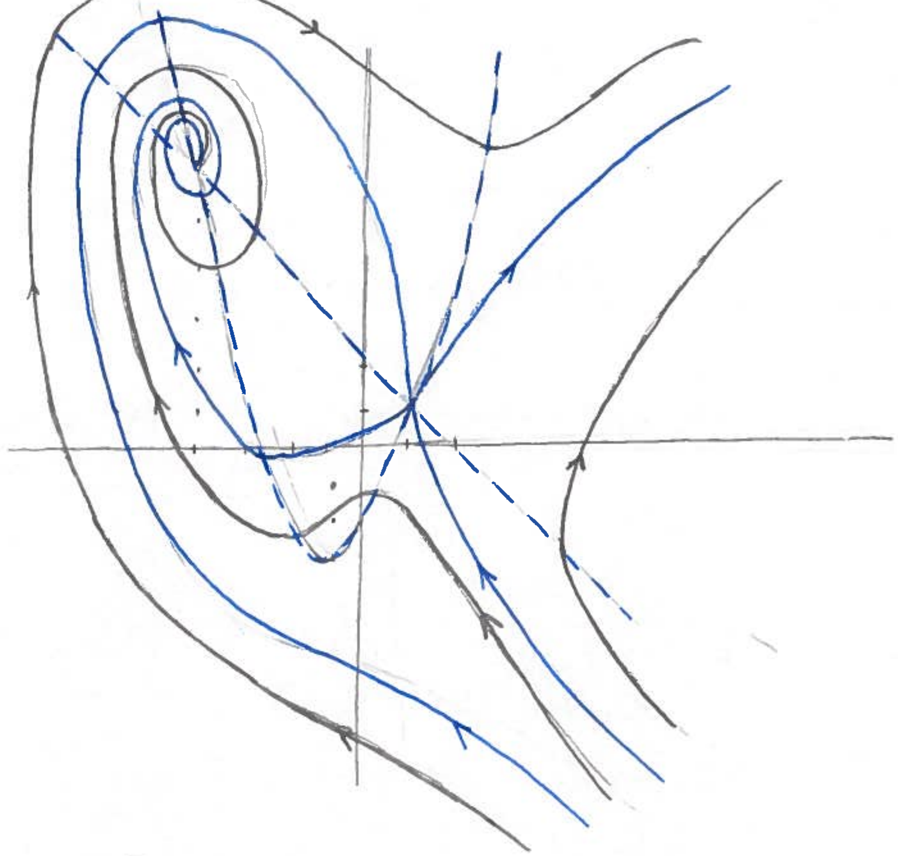
which implies that

$$\begin{aligned}|x^\varepsilon(t) - x^0(t)| &= \frac{e^t}{1+e^t}\frac{|e^t-1|}{1+2\varepsilon^2+e^t}\varepsilon^2 \\ &\leq \frac{e^t}{1+e^t}\frac{1+e^t}{1+e^t}\varepsilon^2 \leq \frac{e^t}{1+e^t}\varepsilon^2 = K(t)\varepsilon^2.\end{aligned}$$

It is also ok to assume that the initial value problem has a unique solution. Exercise E8 and E17 are using the same idea as this one.

Problem 8:
It is not enough to draw the arrows on the isoclines, if you are asked to complete the phase diagram.

4b) STABLE SPIRAL



UNSTABLE SPIRAL