TRAVELING WAVE SOLUTIONS FOR THE KORTEWEG-DE VRIES EQUATION TMA4165

KATRIN GRUNERT

Abstract. The aim of this project is to apply the theory covered in the course TMA4165 to an example. It should illustrate the importance of understanding the connection between the theory and what it means in "reality".

The goal is to study traveling wave solutions, for the Korteweg-de Vries (KdV) equation

$$u_t(t,x) + 6uu_x(t,x) + u_{xxx}(t,x) = 0, \quad t, x \in \mathbb{R}. \tag{1}$$

1. Historical background for the Korteweg-de Vries equation

For the historical background we refer to [1, 2, 3] and the references therein.

2. Traveling wave solutions

Traveling wave solutions are of the form

$$u(t,x) = \phi(x - ct).$$

This means that traveling wave solutions do not change their shape with respect to time, but the wave profile is traveling to the left with speed c.

2.1. Differential equation for ϕ.

As a first step, try to derive an ordinary differential equation for $\phi(y)$. The result will be the second order differential equation

$$-c\phi(y) + 3\phi^2(y) + \phi''(y) = a, \quad a \in \mathbb{R}. \tag{2}$$

2.2. Bounded traveling waves.

The KdV equation models water waves and hence, from a physical point of view, one is interested in identifying those parameters c for which there exist bounded traveling wave solutions. Identify those c.

2.3. Symmetry of bounded traveling waves.

In mathematics a lot of ideas and proofs are much easier if the involved functions are symmetric (not necessarily around 0, but around some point $x = b$). Therefore find out whether or not all bounded traveling wave solutions are symmetric.

Date: December 18, 2017.

2.4. **Bounded non-periodic traveling waves.** Determine whether or not there exist bounded non-periodic traveling wave solutions. If there exist bounded non-periodic traveling wave solutions, compute the one which satisfies $\phi(y) \to 0$ as $y \to \pm\infty$. (The choice of 0 is motivated by the idea that 0 should correspond to a flat water surface.)

If you have done everything right, then you just computed the famous one soliton solution of the KdV equation. There exist a lot of videos of wave channels, where these solutions have been generated. Here is one: https://www.youtube.com/watch?v=w-oDnvbV8mY

2.5. **Bounded periodic traveling waves.** Determine whether or not there exist bounded periodic traveling wave solutions. If there exist bounded periodic traveling wave solutions, compute the ones which satisfy $\min(\phi(y)) = 0$. Moreover, derive an upper and a lower bound for the period of these traveling wave solutions.

If you have done everything right, then you just had a look at the famous cnoidal waves of the KdV equations. There exist a lot of videos of wave channels, where these solutions have been generated. Her is one: https://www.youtube.com/watch?v=8cJo1roykG0.

2.6. **Connection between periodic and non-periodic traveling waves.** Find the connection between the bounds for the period of bounded periodic traveling wave solutions, which satisfy $\min(\phi(y)) = 0$ and the bounded non-periodic traveling wave solution, which satisfies $\phi(y) \to 0$ as $y \to \pm\infty$.

Hint: A non-periodic function, can be seen as a function with period ∞.

References

[1] O. Darrigol. The Spirited Horse, the Engineer, and the Mathematician: Water Waves in Nineteenth-Century Hydrodynamics. Arch. Hist. Exact Sci. **58**, 21–95 (2003).

[2] E. M. de Jager. On the origin of the Korteweg-de Vries equation. https://arxiv.org/abs/math/0602661

[3] J. W. Miles. The Korteweg-de Vries equation: a historical essay. J. Fluid Mech. **106**, 131–147 (1981).