Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4165 Differential Equations and Dynamical Systems
Spring 2018

Exercise set 1

a) You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J. S.: 1.1 (i)–(iii).

b) These exercises will be supervised / discussed in the exercise class:

E1 Aim: Find the equilibrium points and sketch the phase diagram for $\ddot{x} + 3\dot{x} + 2x = 0$.
Idea: Use methods from linear algebra to understand the dynamics of the above equation.
Background: A (real) 2×2 matrix A is diagonalizable, if there exists an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. If A has 2 real eigenvalues λ_1 and λ_2 such that either

- $\lambda_1 \neq \lambda_2$ or
- $\lambda_1 = \lambda_2$ and there exist 2 linearly independent eigenvectors to λ_1,

this is always possible. Given a system of ODEs

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{1}$$

where $\mathbf{x}$ denotes a vector valued function from $\mathbb{R}$ to $\mathbb{R}^2$ and A a diagonalizable matrix, it can be written as $\dot{\mathbf{x}} = PDP^{-1}\mathbf{x}$. Multiplying with P^{-1} from the left and introducing $\mathbf{z} = P^{-1}\mathbf{x}$, we obtain that $\mathbf{z}$ solves

$$\dot{\mathbf{z}} = D\mathbf{z}. \tag{2}$$

a) Rewrite $\ddot{x} + 3\dot{x} + 2x = 0$ as a system of differential equations of first order, i.e. as in (1).

b) Find a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

c) Find the equilibrium points and sketch the phase diagram for $\dot{\mathbf{z}} = D\mathbf{z}$.

d) Find the equilibrium points and sketch the phase diagram for $\dot{\mathbf{x}} = A\mathbf{x}$, with the help of c). Find the connections between the two phase diagrams.

E2 Aim: Sketch the phase diagram for $\ddot{x} = -x - \alpha x^3$ with $\alpha \in \mathbb{R}$.
Idea: Use the connection between computing phase paths and drawing phase diagrams to check that your result is correct.
Background: Consider the differential equation

$$\dot{x} = \alpha x, \quad x(0) = C, \quad \alpha \in \mathbb{R}. \tag{3}$$

Then the corresponding solutions is given by

$$x(t) = Ce^{\alpha t},$$

and in particular, one observes that the behavior of the solution for $t \to \pm\infty$ depends heavily on the value of α.

a) Investigate and sketch the function $f(x) = -x - \alpha x^3$ for different values of α.

b) Rewrite $\ddot{x} = -x - \alpha x^3$ as a system of differential equations of first order.

c) Find the equilibrium points and sketch the phase diagram.

d) Classify all the equilibrium points (with the help of phase paths). Find the connection between the phase diagram close to the equilibrium points and the classification.

E3 Aim: Determine if $2y^2 = 1 - 2x$ is a separatrix for the equation $\ddot{x} + \dot{x}^2 + x = 0$.
Idea: Sketch the phase diagram and identify the curve $2y^2 = 1 - 2x$.

a) Rewrite $\ddot{x} + \dot{x}^2 + x = 0$ as a system of differential equations of first order.

b) Compute the corresponding phase paths.

c) Sketch the phase diagram.

d) Is $2y^2 = 1 - 2x$ a separatrix or not?