Norwegian University of Science
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TMA4165 Differential Equations and Dynamical Systems Spring 2018

Exercise set 11

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J.S.: 11.5

Exam 1996, 1 Given the system

$$\dot{x} = x - y$$
$$\dot{y} = x^2 - 1.$$

a) Find and classify all equilibrium points of the system. Sketch the phase diagram.

b) Does there exist a closed phase path surrounding all equilibrium points?

Exam 1996, 6 Compute the index of the origin for the following systems

a)

$$\dot{x} = x$$
$$\dot{y} = -y.$$

b)

$$\dot{x} = x + x^4 + y^5$$
$$\dot{y} = -y + xy^3.$$

Exam 2002, 3 **a)** State Bendixson's negative criterion.

b) Determine whether or not the following system has non-constant periodic solutions.

$$\dot{x} = y$$
$$\dot{y} = -x - y(1 + x^2 + x^4).$$

c) Given the population model

$$\dot{x} = xF(x, y)$$
$$\dot{y} = yG(x, y),$$

where F and G are C^1 functions. Assume that $\frac{\partial F}{\partial x} < 0$ and $\frac{\partial G}{\partial y} < 0$. Show that there are no closed phase paths in the first quadrant.

These exercises will be supervised / discussed in the exercise class:

E31 **a)** Show that the system

$$\dot{x} = x - y - x^3$$
$$\dot{y} = x + y - y^3$$

has a closed phase path inside the region

$$A_{a,b} = \{(x, y) \mid a \leq x^2 + y^2 \leq b, 0 < a < 1, b > 2\}.$$

b) Consider the system in a) for the region $A_{\frac{3}{4},3}$. Explain why the result in a) does not contradict Bendixson's negative criterion.

E32 Given the system

$$\dot{x} = x + y - x\sqrt{x^2 + y^2} \tag{1}$$
$$\dot{y} = -x + y - y\sqrt{x^2 + y^2}.$$

a) Classify the equilibrium point $(0, 0)$ for both (1) and its linearisation.

b) Show that the system has exactly one closed phase path.

c) Define what it means to be a Poincaré map with Poincaré section Σ.

d) Determine the Poincaré map with Poincaré section $\Sigma = \{(x, 0) \mid x > 0\}$.

E33 **a)** Given the autonomous two-dimensional system $\dot{x} = f(x)$, where $f : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is a Lipschitz function.
Explain which ω-limit sets a phase path Γ can have if Γ lies inside a closed, bounded subset K of $\mathbb{R}^2$.

b) Given the following systems in polar coordinates

$$\dot{r} = (1 - r^2)r^2 \tag{2a}$$
$$\dot{\theta} = 1 \tag{2b}$$

and

$$\dot{r} = (1 - r^2)^2 r \tag{3a}$$
$$\dot{\theta} = 1 - r^2 \tag{3b}$$

Find and classifiy all possible ω-limit sets and determine whether they are stable or unstable.

E34 Aim: Understand the connection between the index of an equilibrium point and the phase diagram close to this equilibrium point.
Sketch an example of a phase diagram around an equilibrium point of index -2, 1, and 3, respectively.