Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4165 Differential Equations and Dynamical Systems Spring 2018

Exercise set 12

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J.S: 11.8, 11.9, 11.10, 12.1 (ii)

Exam 1992, 3 Give an example of an n-dimensional, dynamical system (n given and $n \geq 2$)

$$\dot{x} = f(x), \quad x \in \mathbb{R}^n$$

such that $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, $f(0) = 0$, $\lim_{t\to\infty} x(t) = 0$ for all solutions, and not all eigenvalues of its linearisation at 0 have strictly negative real part.

These exercises will be supervised / discussed in the exercise class:

E35 Aim: Investigate the bifurcation of the system

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= y^2 - \lambda,\end{aligned}$$

at $\lambda = 0$.

a) Sketch the phase diagram for $\alpha < 0$, $\alpha = 0$, and $\alpha > 0$.

b) Draw a stability diagram in that (λ, y) plane for $x = 0$. What type of bifurcation occurs at $\lambda = 0$?

E36 Aim: Investigate the bifurcation of the system

$$\begin{aligned}\dot{x} &= \mu x - x^2,\\ \dot{y} &= y(\mu - 2x).\end{aligned}$$

a) Find and classify all equilibrium points of the system.

b) Obtain the equations of the phase paths and sketch the phase diagrams for different values of μ.

c) Draw a stability diagram in the (μ, x) plane for $y = 0$. Find the bifurcation point and determine what type of bifurcation occurs.

E37 Aim: Investigate the bifurcation of the following system in polar coordinates

$$\dot{r} = r(r^2 - \mu r + 1)$$
$$\dot{\theta} = -1,$$

at $\mu = 2$.

a) Sketch the phase diagram for $\mu < 2$, $\mu = 2$, and $\mu > 2$.

b) What type of bifurcation occurs at $\mu = 2$?

E38 Aim: Investigate the bifurcation of the system

$$\dot{x} = \lambda x + y \tag{1a}$$
$$\dot{y} = x - x^2, \tag{1b}$$

at $\lambda = 0$.

a) Show that the system (1) for $\lambda = 0$ has a homoclinic phase path by finding its equation.

b) Sketch the phase diagram for $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. What happens to the homoclinic phase path from a)?

Exam 2014, 5 **a)** State the Poincaré Bendixson theorem.

b) Let $\dot{x} = f(x)$ and $\dot{x} = g(x)$ be two systems in $\mathbb{R}^2$, where f and g are C^1 functions such that $\langle f(x), g(x) \rangle = 0$. Show that if $\dot{x} = f(x)$ has a periodic solution, then the system $\dot{x} = g(x)$ has at least one equilibriumpoint.

Exam 1995, 6 Let $f : E \mapsto \mathbb{R}^2$ be a C^1 vector field and $E \subset \mathbb{R}^2$ open, such that there exists an annulus A with $A \subset E$. f has no zeros inside A or on the boundary of A, and f points inwards along the boundary of A. Why must A contain at least one closed phase path? Show that if A contains 3 closed phase paths, then at least one of them must be a stable limit cycle.

Exam 2002, 4 Given the dynamical system $\dot{x} = f(x)$, where f belongs to C^1. Let

$$A = \{x \in \mathbb{R}^2 \mid 1 \leq \|x\| \leq 2\}.$$

Assume that $f(x) \neq 0$ for all x on the boundary of A. Sketch all possible phase diagrams in A under the assumptions that there are neither equilibrium points nor closed phase paths inside A, that the boundaries of A are closed phase paths, and that the boundaries of A either have the same or opposite orientation.