

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:

J. S.: 1.4, 1.6.

Some explanation to 1.6: The potential energy $\mathcal{V}(x)$ of a conservative system ..., means that you consider the problem $\ddot{x} = f(x)$ where $\mathcal{V}(x) = -\int f dx$ and $\mathcal{V}(x)$ as given in the text.

These exercises will be supervised / discussed in the exercise class:

- E4 Aim: Sketch the phase diagram for the equation $\ddot{x} e^x = a, a \in \mathbb{R}$.
 - **a)** Investigate and sketch the function $f(x) = e^x + a$ for different values of a
 - **b)** Rewrite $\ddot{x} e^x = a$ as a system of differential equations of first order.
 - c) Find the equilibrium points and sketch the phase diagram.
 - d) Mark the separatrix in the phase diagram.
- E5 Aim: Show that the initial value problem

$$\dot{x} = \frac{1}{2}x^2 + \frac{1}{2}|x|^3, \quad x(0) = 1$$
 (1)

has exactly one local solution, that $\lim_{t\to-\infty} x(t) = 0$ and that there exists $a \in (\frac{1}{2}, 1)$ such that $\lim_{t\to a^-} x(t) = \infty$.

Idea: Estimate the maximal interval of existence for the above initial value problem by using a comparison argument.

Background: Given two continuously differentiable functions $y : \mathbb{R} \to \mathbb{R}$ and $z : \mathbb{R} \to \mathbb{R}$ such that y(0) = z(0). Then one has that $\dot{y}(t) \leq \dot{z}(t)$ for all $t \in \mathbb{R}$ implies that

$$y(t) - y(0) = \int_0^t \dot{y}(s) ds \le \int_0^t \dot{z}(s) ds = z(t) - z(0)$$
 for all $t \ge 0$,

which is equivalent to $y(t) \leq z(t)$ for all $t \geq 0$. Similarly it follows that $z(t) \leq y(t)$ for all $t \leq 0$.

a) Show that the functions $f(x) = x^2$ and $g(x) = |x|^3$ are locally Lipschitz continuous and conclude that (1) has a unique local solution.

- b) Find the unique solution to the initial value problem $\dot{x} = f(x)$, $x(0) = x_0$ for $x_0 < 0, x_0 = 0, x_0 > 0$.
- c) Find the unique solution to the initial value problem $\dot{x} = g(x)$, $x(0) = x_0$ for $x_0 < 0, x_0 = 0, x_0 > 0$.
- **d)** Use a comparison argument based on the solutions found in b) and c) to show that x(t), the solution of (1), satisfies that $\lim_{t\to-\infty} x(t) = 0$ and that there exists $a \in (\frac{1}{2}, 1)$ such that $\lim_{t\to a^-} x(t) = \infty$. Note, this means in particular that the maximal interval of existence of x(t) is given by $(-\infty, a)$.

E6 Aim: sketch the phase diagram for the linear system:

$$\dot{x} = -4x + 2y \tag{2}$$
$$\dot{y} = -3x + y.$$

Background: Given some initial value problem $\dot{x}(t) = f(x(t)), x(0) = a$, then y(t) = x(-t) solves the initial value problem $\dot{y}(t) = -f(y(t)), y(0) = a$. That means in particular, if $x(t) \to \infty$ as $t \to \infty$ then $y(t) \to \infty$ as $t \to -\infty$ and if $x(t) \to 0$ as $t \to -\infty$ then $y(t) \to 0$ as $t \to \infty$.

- a) Define $z_1(t) = x(-t)$ and $z_2(t) = y(-t)$. Find the linear system for $z_1(t)$ and $z_2(t)$.
- **b)** Sketch the phase diagram for the system for $z_1(t)$ and $z_2(t)$.
- c) Sketch the phase diagram for (2).
- d) Compare the two diagrams obtained in b) and c).