Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4165 Differential Equations and Dynamical Systems
Spring 2018

Exercise set 4

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J. S.: 2.3 (i), (vi), (ix)
Ex 2013.2

These exercises will be supervised / discussed in the exercise class:

E10 Find the Hamiltonian function for the system

$$\begin{aligned}\dot{x} &= y\\ \dot{y} &= -x + x^3.\end{aligned}$$

and sketch the phase portrait for this system.

E11 Aim: Find a necessary condition for an equilibrium point to be a centre.

a) Let $f,\ g \in C^1(\mathbb{R})$ such that

- f has a local minimum at $x = a$ and f' is strictly increasing locally around a and
- g has a local minimum at $y = b$ and g' is strictly increasing locally around b.

Show that the function $h(x,y) = f(x)+g(y)$ has a local minimum in (a,b). Furthermore deduce that there exists a neighborhood of (a,b) in which all solutions of the family of equations

$$f(x) + g(y) = \mathit{constant}$$

represent closed curves surrounding (a,b).

b) Show that $(0,0)$ is a centre for the system

$$\begin{aligned}\dot{x} &= y^5\\ \dot{y} &= -x^3\end{aligned} \tag{1}$$

and that all paths are closed curves by computing the phase paths. Identify the functions f and g from a) for the system (1).

c) Given the system

$$\dot{x} = F(y) \tag{2a}$$
$$\dot{y} = -G(x). \tag{2b}$$

Find necessary conditions on F and G such that an equilibrium point (a, b) of (2) is a centre by applying a).

$\boxed{\text{E12}}$ Aim: Show that the origin is a spiral point of the system

$$\dot{x} = -y - x\sqrt{x^2 + y^2} \tag{3a}$$
$$\dot{y} = x - y\sqrt{x^2 + y^2}, \tag{3b}$$

but a centre for its linear approximation.

a) Find the linear approximation to (3) and show that the origin is a centre.

b) Show that the origin is a spiral for (3).