Norwegian University of Science and Technology
Institutt for matematiske fag

TMA4165 Differential Equations and Dynamical Systems
Spring 2018

Exercise set 6

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J.S.: 8.11, 8.14
Exam 2016.2 (A homoclinic phase path is a phase path that connects an equilibrium point with itself.)

A1 Determine whether the following pairs of solutions to $\dot{\mathbf{x}} = A\mathbf{x}$ are linearly independent or not, for $t \in [0, \infty)$.

a)

$$\mathbf{x_1} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

b)

$$\mathbf{x_1} = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} 2e^t \\ 4e^t \end{pmatrix}$$

c)

$$\mathbf{x_1} = \begin{pmatrix} e^t \\ 2e^t \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} te^t \\ (2t+1)e^t \end{pmatrix}$$

d) Deduce from the given solutions in c), how the corresponding system $\dot{\mathbf{x}} = A\mathbf{x}$ looks like. What can you say about the corresponding eigenvalues and eigenvectors from just looking at $\mathbf{x}_1$ and $\mathbf{x}_2$?

e) Is the following pair of functions linearly independent?

$$\mathbf{x_1} = \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} te^t \\ (2t+1)e^t \end{pmatrix}.$$

f) Is the following pair of functions linearly independent?

$$\mathbf{x_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{x_2} = \begin{pmatrix} t \\ t \end{pmatrix}.$$

These exercises will be supervised / discussed in the exercise class:

E16 Given the system

$$\dot{\mathbf{x}} = A(t)\mathbf{x},$$

where $A(t) \in \mathbb{R}^{n\times n}$, $\mathbf{x}(t) \in \mathbb{R}^n$. for all t. Assume that the solutions to the above system are unique. Given $m \leq n$ and m linearly independent vectors b_i, $i = 1, \ldots, m$, show that the functions $\phi_i(t)$, $i = 1, \ldots, m$, which solve

$$\dot{\phi}_i(t) = A(t)\phi_i(t), \quad \phi_i(0) = b_i,$$

are linearly independent.

E17 Aim: Given the system $\dot{\mathbf{x}} = A\mathbf{x}$ for $A \in \mathbb{R}^{2\times 2}$ and $\mathbf{x} \in \mathbb{R}^2$. Assume that $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are two solutions which satisfy $|\mathbf{x}_1(0) - \mathbf{x}_2(0)| = \varepsilon$. Show that

$$\varepsilon e^{-K|t|} \leq |\mathbf{x}_1(t) - \mathbf{x}_2(t)| \leq \varepsilon e^{K|t|}$$

for some $K > 0$.

a) Which system does $\mathbf{y}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$ solve?

b) Show that $-2K|\mathbf{y}(t)|^2 \leq \frac{d}{dt}|\mathbf{y}(t)|^2 \leq 2K|\mathbf{y}(t)|^2$ for some $K > 0$.

c) Show that $\varepsilon e^{-K|t|} \leq |\mathbf{y}(t)| \leq \varepsilon e^{K|t|}$.

E18 Aim: Let $\mathbf{x_1}$ and $\mathbf{x_2}$ respectively solve the initial value problems

$$\begin{aligned} \dot{\mathbf{x}}_1 &= A_1(t)\mathbf{x_1}, \quad \mathbf{x}_1(0) = a \\ \dot{\mathbf{x}}_2 &= A_2(t)\mathbf{x_2}, \quad \mathbf{x}_2(0) = a. \end{aligned}$$

Show, under the assumption that $\max_{t\in[0,T]} (\|A_1(t)\|, \|A_2(t)\|) \leq K_T$, that for all $t \in [0, T]$,

$$|\mathbf{x}_1(t) - \mathbf{x}_2(t)| \leq M_T C_T \max_{t\in[0,T]} \|A_1(t) - A_2(t)\|.$$

Here $\max_{t\in[0,T]} (|\mathbf{x}_1(t)|, |\mathbf{x}_2(t)|) \leq M_T$, and

$$C_T \leq \sqrt{\frac{e^{(2K_T+1)T} - 1}{2K_T + 1}}.$$

a) Show that for $i = 1, 2$, there exists M_T such that $|\mathbf{x_i}(t)| \leq M_T$ for all $t \in [0, T]$ under the assumption that $\|A_i(t)\| \leq K_T$ for all $t \in [0, T]$.

b) Let $\mathbf{y}(t) = \mathbf{x}_1(t) - \mathbf{x}_2(t)$. Show that for all $t \in [0, T]$,

$$\frac{d}{dt}|\mathbf{y}(t)|^2 \leq (1 + 2K_T)|\mathbf{y}(t)|^2 + M_T^2 \max_{t\in[0,T]} \|A_1(t) - A_2(t)\|^2.$$

c) Show that for all $t \in [0, T]$,

$$|\mathbf{y}(t)| \leq M_T C_T \max_{t\in[0,T]} \|A_1(t) - A_2(t)\|.$$