Norwegian University of Science and Technology
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# TMA4165 Differential Equations and Dynamical Systems Spring 2018

**Exercise set 7**

**You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:**
Ex 2013.5

Ex 1993, 1 Given the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Find, with the help of the matrix exponential, the solution which satisfies $x(0) = y(0) = z(0) = 1$.

Ex 1995, 5 Given the linear system

$$\begin{aligned} \dot{x} &= 3x - 2y \\ \dot{y} &= 2x + 3y. \end{aligned}$$

**a)** Consider the solution $(x(t), y(t))$ which satisfies $(x(0), y(0)) = (x_0, 0)$ where $x_0 > 0$. Denote by $t_1 > 0$ the first time the solution crosses the positive $x$-axis. Show that $(x(t_1), y(t_1)) = (x_0 e^{3\pi}, 0)$.

**b)** Classify the equilibrium point $(0, 0)$ of the given system.
Classify the equilibrium point $(0, 0)$ of the following system

$$\begin{aligned} \dot{x} &= 3x - 2y + (x^2 + y^2)^2, \\ \dot{y} &= 2x + 3y + x^3 + y^4. \end{aligned}$$

Ex 1992, 2 Given the system

$$\begin{aligned} \dot{x} &= -x - xy^2 - x^3 \\ \dot{y} &= -7y + 3x^2 y - 2yz^2 - y^3 \\ \dot{z} &= -5z + y^2 z - z^3. \end{aligned}$$

**a)** Show that the origin is an asymptotically stable equilibrium point.

**b)** Show that every solution $(x(t), y(t), z(t))$ tends to the origin as $t \to \infty$.

**These exercises will be supervised / discussed in the exercise class:**

**E19** Aim: Prove the following generalization of the Gronwall inequality presented in the lecture. If, for $t \geq 0$

- $u(t)$ and $g(t)$ are continuous and $g(t) \geq 0$ and $u(t) \geq 0$,
- $f(t)$ is continuous, non-decreasing and $f(t) > 0$
- $$u(t) \leq f(t) + \int_0^t g(s)u(s)ds, \tag{1}$$

then
$$u(t) \leq f(t) \exp\left(\int_0^t g(s)ds\right). \tag{2}$$

Background: We proved in the lecture the following lemma: If, for $t \geq 0$

- $w(t)$ and $v(t)$ are continuous and $w(t) \geq 0$ and $v(t) \geq 0$,
- $$w(t) \leq K + \int_0^t w(s)v(s)ds, \quad K > 0, \tag{3}$$

then
$$w(t) \leq K \exp\left(\int_0^t v(s)ds\right). \tag{4}$$

**a)** Rewrite equation (1) in such a way that it is of the form (3).

**b)** Use the Gronwall inequality stated in the background to prove (2).

**E20** Aim: Use the series definition of $e^A$ to prove some properties of the exponential function of a matrix $A$.

**a)** Show that $e^{A+B} = e^A e^B$ if $AB{=}BA$.

**b)** Find $2 \times 2$ matrices $A$ and $B$ such that $e^{A+B} \neq e^A e^B$.

**c)** Show that $e^A$ is nonsingular and $(e^A)^{-1} = e^{-A}$.

**d)** Show that $(e^A)^T = e^{A^T}$.

**e)** Show that $\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$.

**E21** Let $A \in \mathbb{R}^{n\times n}$, $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{x} : \mathbb{R} \mapsto \mathbb{R}^n$ be a solution of

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}.$$

**a)** Define what it means for $\mathbf{x}(t)$ to be (Liapunov) stable.

**b)** Show that all solutions of the above equation are (Liapunov) stable if there is a fundamental matrix $\Phi : \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ for $\dot{\mathbf{x}} = A\mathbf{x}$ such that

$$\|\Phi(t)\| \leq C < \infty \quad \text{for all} \quad t \geq 0.$$