TMA4165 Differential Equations and Dynamical Systems
Spring 2018

Norwegian University of Science and Technology
Institutt for matematiske fag

Exercise set 9

You find solutions to the following exercises on the web page. Give it a try and ask if something is unclear:
J.S.: 1.16, 3.1, 3.3

Exam 1995,1

a) Determine if the following system is stable or unstable at the origin

$$\dot{x} = e^{-x-3y} - 1$$
$$\dot{y} = x(1 - y^2).$$

b) Given the system

$$\dot{x} = x - y$$
$$\dot{y} = 1 - xy.$$

Find and classify all equilibrium points of the system. Sketch the phase diagram, with orientations.

These exercises will be supervised / discussed in the exercise class:

E25 Aim: Find the homoclinic paths of

$$\ddot{x} - x + 3x^5 = 0. \tag{1}$$

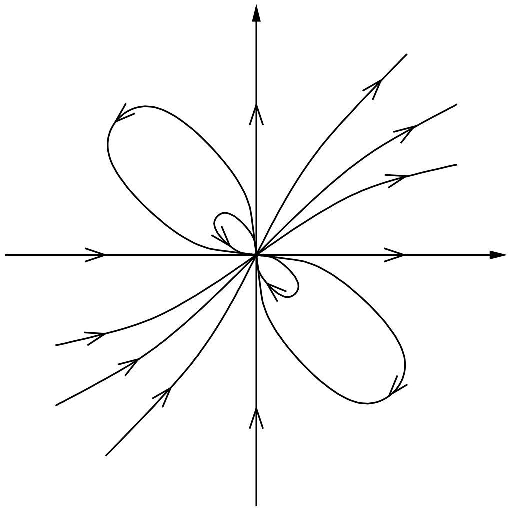
a) Find and classify all equilibrium points of (1) and sketch the phase diagram.

b) Compute the phase paths of (1).

c) Derive the solutions $x_1(t)$, $x_2(t)$ which correspond to the homoclinic paths and satisfy $x_1(t_0) = -1$ and $x_2(t_0) = 1$, respectively.
Hint: Derive, using the equation for the homoclinic phase paths from b), the second order differential equation for $z(t) = \frac{1}{x(t)^2}$ for each of the two homoclinic paths.
(This ansatz seems reasonable, since the homoclinic paths satisfy $\frac{1}{x^4} - \frac{y^2}{x^6} = 1$, which is quite similar to $\cosh^2(t) - \sinh^2(t) = 1$.)

E26 Aim: Compute the index of two-dimensional systems in different ways.

a) Find the index of the origin given the following phase diagram.



b) Let $z = x + iy$ $(x, y \in \mathbb{R})$. Given the following dynamical systems in the complex plane

$$\dot{z} = z^k \quad \text{and} \quad \dot{z} = \bar{z}^k, \tag{2}$$

show that the index of the origin equals k and $-k$, respectively.

E27 Aim: Find out as much as possible about the system

$$\dot{x} = y^2 - x^2$$
$$\dot{y} = 1 + 2xy.$$

a) Find and classify all equilibrium points of the system.

b) Determine whether or not the above system has non-constant periodic solutions.

c) Show that the given system is Hamiltonian and find a Hamiltonian function for the system.

d) Sketch the phase diagram and show that the phase path through the origin satisfies

$$x = \frac{2y^3}{3(1 + \sqrt{1 + \frac{4}{3}y^4})}.$$