$$\dot{\vec{x}} = A(t)\vec{x} + f(t)$$

($A(t)$.. $n \times n$ MATRIX
ENTRIES FUNCTIONS OF t)

$\vec{x}_1(t), \vec{x}_2(t)$ SOLUTIONS

$$\vec{z}(t) = \vec{x}_1(t) - \vec{x}_2(t)$$

SOLVES

$$\dot{\vec{z}}(t) = A(t)\vec{z}(t)$$

ALL SOLUTIONS TO
$\dot{\vec{x}} = A(t)\vec{x} + f(t)$
HAVE THE SAME
STABILITY AS THE
ZERO SOLUTION
TO $\dot{\vec{z}} = A(t)\vec{z}$

$\vec{x}_p(t)$ SOL TO $\dot{\vec{x}} = A(t)\vec{x} + f(t)$

EVERY SOLUTION TO
$\dot{\vec{x}} = A(t)\vec{x} + f(t)$
IS OF THE FORM
$\vec{x}(t) = \vec{x}_p(t) + \vec{\phi}_c(t)$
($\vec{\phi}_c(t)$ SOL. TO $\dot{\vec{z}} = A(t)\vec{z}$)

$\Phi(t)$ FUNDAMENTALMATRIX
TO $\dot{\vec{z}} = A(t)\vec{z}$

$$\vec{x}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{x}_0 + \Phi(t)\int_{t_0}^{t} \Phi^{-1}(s) f(s)\,ds$$

SOLVES: $\dot{\vec{x}} = A(t)\vec{x} + f(t)$, $\vec{x}(t_0) = \vec{x}_0$

$A(t) = A$

$$\vec{x}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{x}_0 + \int_{t_0}^{t} \Phi(t-s+t_0)\Phi^{-1}(t_0) f(s)\,ds$$

SOLVES $\dot{\vec{x}} = A\vec{x} + f(t)$, $\vec{x}(t_0) = \vec{x}_0$

$$\dot{\vec{x}} = A\vec{x}, \quad \vec{x}(t_0) = \vec{x}_0$$
(A $n \times n$ MATRIX)

PICARD ITERATION

$\exists n$ LIN IND SOL
$\phi_1(t), \dots, \phi_n(t)$
TO $\dot{\vec{x}} = A\vec{x}$

$$\xrightarrow{\Phi(t) = [\phi_1(t), \dots, \phi_n(t)]}$$

$$\vec{x}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{x}_0$$

$x_0(t) = \vec{x}_0 \quad \forall t$

$$x_1(t) = \vec{x}_0 + \int_{t_0}^{t} A\vec{x}_0(s)\,ds = (I + A(t-t_0))\vec{x}_0$$

$$\vec{x}_n(t) = \vec{x}_0 + \int_{t_0}^{t} A\vec{x}_{n-1}(s)\,ds = \sum_{k=0}^{n} \frac{A^k(t-t_0)^k}{k!}\vec{x}_0$$

$$\Big\downarrow {\scriptstyle n \to \infty}$$

$$\vec{x}(t) = \sum_{k=0}^{\infty} \frac{A^k(t-t_0)^k}{k!}\vec{x}_0 = \exp(A(t-t_0))\vec{x}_0$$

$$\Rightarrow \vec{x}(t) = \Phi(t)\Phi^{-1}(t_0)\vec{x}_0 = \exp(A(t-t_0))\vec{x}_0$$

GENERAL: A $n \times n$ MATRIX $\quad e^{A} = \exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}$

- $\|e^{A}\| \leq e^{\|A\|}$
- $e^{\phi} = I$ <u>IF</u> ϕ = ZERO MATRIX
- $e^{A+B} = e^{A}e^{B}$ <u>IF</u> $AB = BA$
- $e^{-A} = (e^{A})^{-1}$
- $\frac{d}{dt} e^{At} = Ae^{At} = e^{At}A$
- $(e^{At})^{T} = e^{A^{T}t}$
- $\lambda_1, \dots, \lambda_n$ EIGENVALUES OF A

  $\Rightarrow \forall \gamma > \max_i \operatorname{Re}(\lambda_i) \ \exists c > 0$ ST $\|e^{At}\| \leq ce^{\gamma t}$