$$\begin{tikzcd}[column sep=small]
 & \begin{array}{c} \dot{\vec{x}} = f(\vec{x}),\ \vec{x} \in \mathbb{R}^2 \\ E = \{\vec{x} \in \mathbb{R}^2 \mid f(\vec{x}) = \vec{0}\} \\ \mathcal{C} \ldots \text{CLOSED CURVE} \end{array} \arrow[dl, "E \cap \overline{int(\mathcal{C})} = \phi"'] \arrow[dr, "\mathcal{C}\ \text{CLOSED PHASE PATH}"] & \\
I_{\mathcal{C}} = 0 \arrow[d] & & I_{\mathcal{C}} = 1 \arrow[d] \\
\mathcal{C} \ldots \text{NO PHASE PATH} & & \exists \vec{e} \in E : \vec{e} \in int(\mathcal{C}).
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{array}{c} \dot{\vec{x}} = f(\vec{x}),\ \vec{x} \in \mathbb{R}^2 \\ E = \{\vec{x} \in \mathbb{R}^2 : f(\vec{x}) = \vec{0}\} \\ \Omega \subseteq \mathbb{R}^2 \mid E \cap \Omega = \phi \end{array} \arrow[d, "\exists \vec{x}_0\ \Omega :\ \mathcal{H}_{\vec{x}_0} \subseteq K \subseteq \Omega\ (K \ldots \text{COMPACT})"] \\
\omega(\mathcal{H}_{\vec{x}_0}) \ldots \text{CLOSED PHASE PATH}
\end{tikzcd}$$

$\dot{\vec{x}} = f(\vec{x}),\ \vec{x}\in\mathbb{R}^2$

$E = \{\vec{x}\in\mathbb{R}^2 \mid f(\vec{x}) = \vec{0}\}$

$\Omega \subseteq \mathbb{R}^2 \quad E\cap\Omega = \emptyset$

$e \in E$

$\exists\ \Omega_S \subseteq \Omega$ COMPACT

$\downarrow$

$\vec{x}_0 \in \Omega_S \Rightarrow \gamma_{\vec{x}_0} \in \Omega_S$

$\downarrow$

$\omega(\gamma_{\vec{x}_0}) \in \Omega_S \quad \forall \vec{x}_0 \in \Omega_S$

$\omega(\gamma_{\vec{x}_0})$ ... CLOSED PHASE PATH

$\downarrow$

$\dot{\vec{x}} = f(\vec{x})$

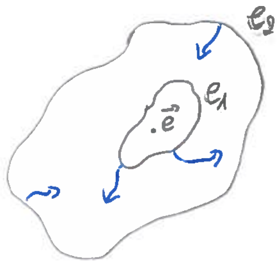
HAS AT LEAST
ONE PERIODIC SOLUTION

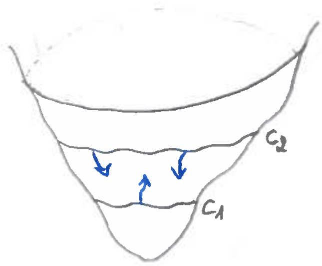
HOW TO FIND Ω_S?

$v: \Omega \subseteq \mathbb{R}^2 \to \mathbb{R}$ CONT

LOCAL STRICT MINIMUM AT $\vec{e}$

LEVEL CURVES = SIMPLE CLOSED CURVES





$$\underbrace{\vec{x}_0 \in \Omega_S \quad \gamma_{\vec{x}_0} \in \Omega_S \qquad v(\vec{x}_0) \in [c_1, c_2] \quad v(\vec{x}(t)) \in [c_1, c_2]}$$

$(v(\vec{x}(t)))^\cdot \le 0$ FOR $\vec{x}(t)$ ST $v(\vec{x}(t)) = c_2$

$(v(\vec{x}(t)))^\cdot \ge 0$ FOR $\vec{x}(t)$ ST $v(\vec{x}(t)) = c_1$

$$\ddot{x} + f(x,\dot{x})\dot{x} + g(x) = 0$$

$\downarrow$

$$\dot{x} = y$$
$$\dot{y} = -f(x,y)y - g(x)$$

$\downarrow$ $g(x) \begin{cases} <0 & x<0 \\ =0 & x=0 \\ >0 & x>0 \end{cases}$

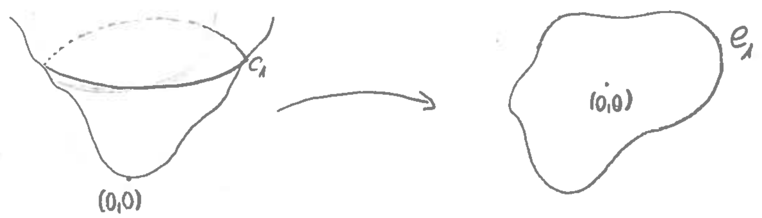
EQUILIBRIUM POINT $(0,0)$

$\downarrow$ INDEX THEORY

EVERY CLOSED PHASE PATH
<u>MUST</u> SURROUND $(0,0)$

$\downarrow$ $G(x) = \int_0^x g(u)\,du \to \infty \quad x \to \pm\infty$

$$v(x,y) = \frac{1}{2}y^2 + G(x)$$



$\downarrow$

$$(v(x,y))^{\cdot} = -y^2 f(x,y)$$

$f(0,0) < 0$
f CONT

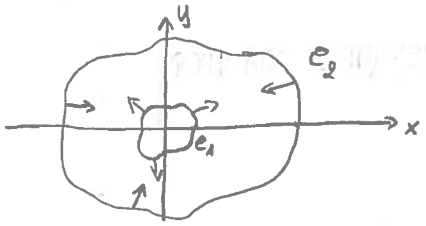
$\downarrow$

$\exists$ LEVEL CURVE C_1 ST
$(v(x,y))^{\cdot} \geq 0 \quad \forall (x,y) \in C_1$

$\exists a > 0$ ST $f(x,y) > 0 \quad \forall (x,y)$ ST $x^2+y^2 > a^2$

$\downarrow$

$\exists$ LEVEL CURVE C_2 ST
$(v(x,y))^{\cdot} \leq 0 \quad \forall (x,y) \in C_2$ AND $C_2 \cap \{(x,y) : x^2+y^2 \leq a^2\} = \emptyset$



$\downarrow$ POINCARÉ BENDIXSON

$\exists$ A CLOSED PHASE PATH
$\triangleq \exists$ PERIODIC SOLUTION